Time domain analysis of a transmission problem between the vacuum and a metamaterial in electromagnetism

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Maxwell's equations in dispersive media

Maxwell's equations unknowns :

(E, H)	electric / magnetic fields
(D, B)	electric / magnetic inductions

$$\partial_t D - \text{rot } H = 0,$$

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Metamaterials : $\varepsilon(\omega)$ and / or $\mu(\omega)$ become real negative in some range of frequencies $\Longrightarrow$ many applications in optics.

Maxwell's equations in dispersive media

$$\begin{cases} \partial_t D - \text{rot } H = 0, \\ \\ \partial_t B + \text{rot } E = 0, \end{cases} \qquad D = \varepsilon_0 \big(E + \Omega_e^2 \, P \big) \qquad B = \mu_0 \, \big(H + \Omega_m^2 \, M \big)$$

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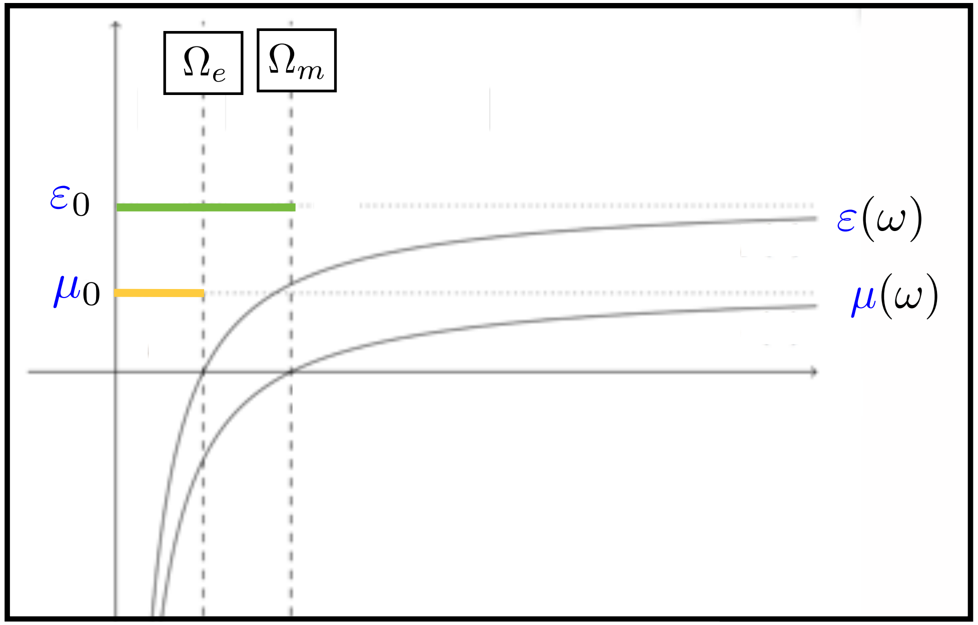
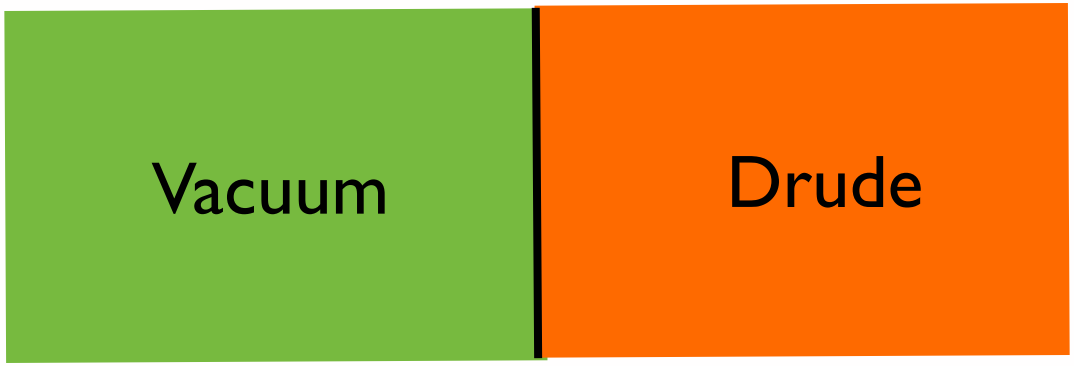
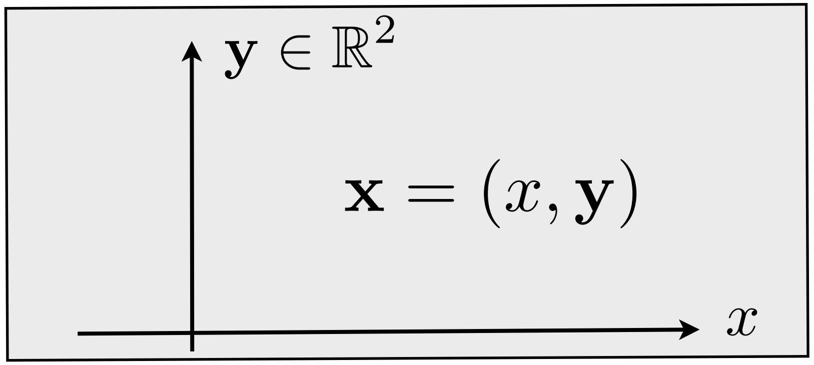


The time domain transmission problem





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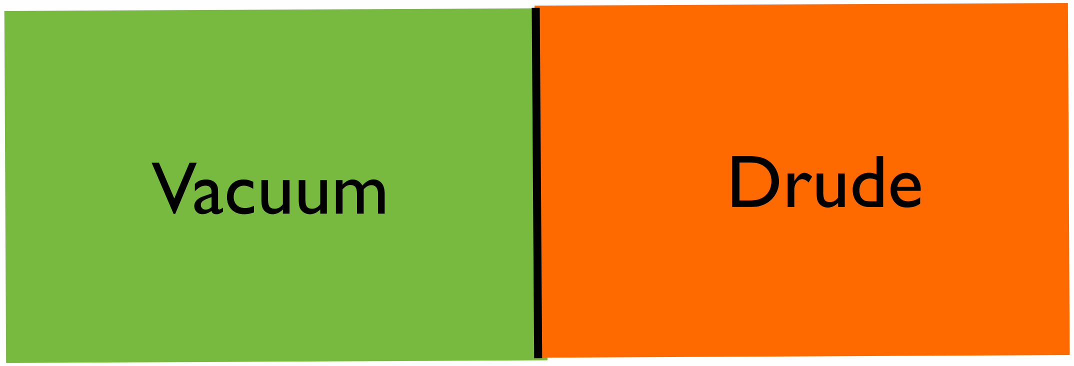

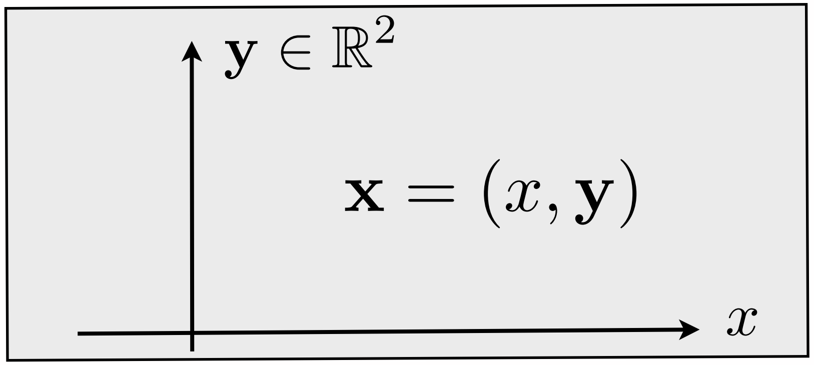

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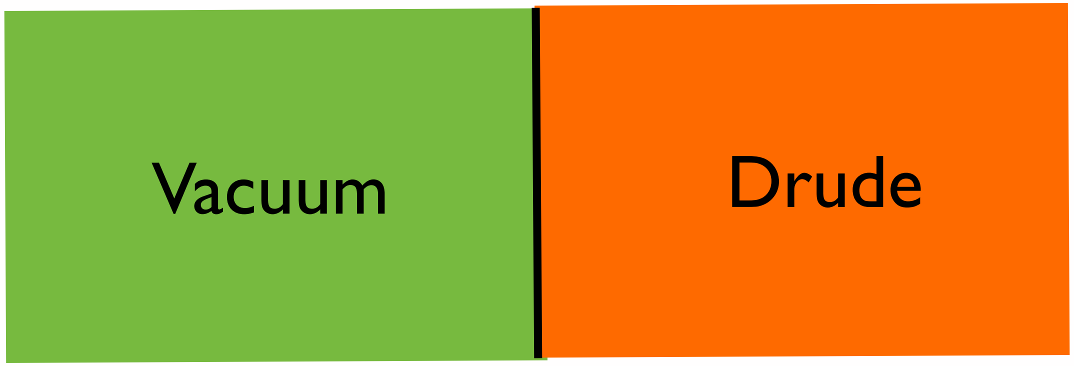

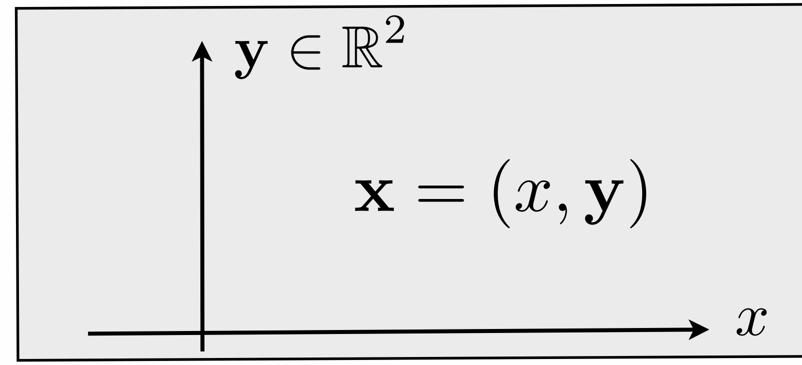

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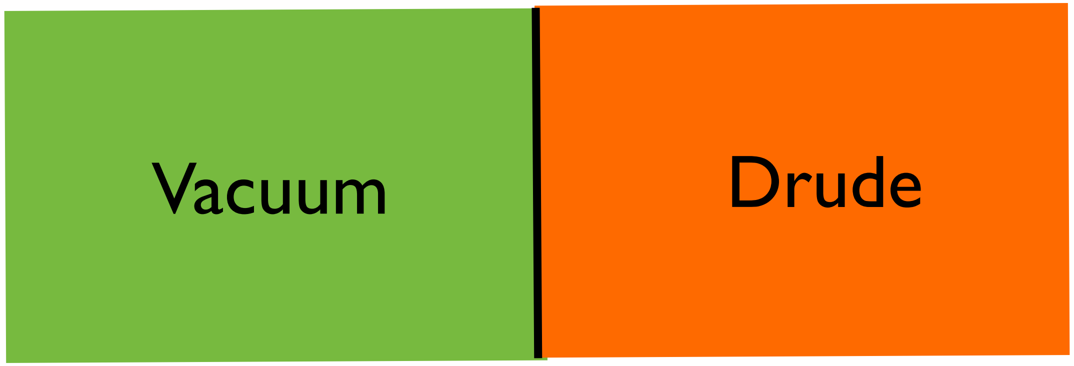

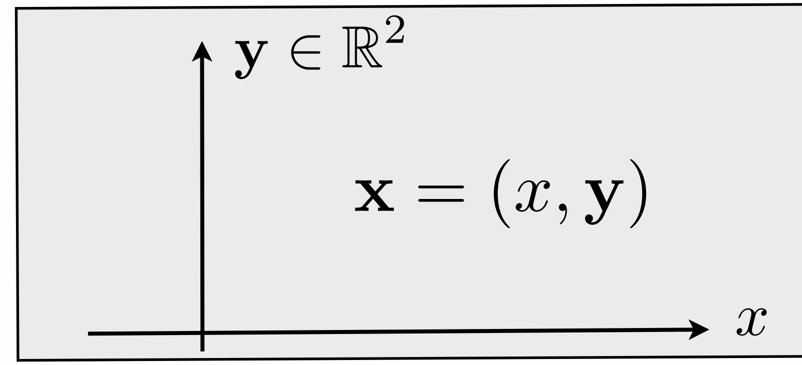

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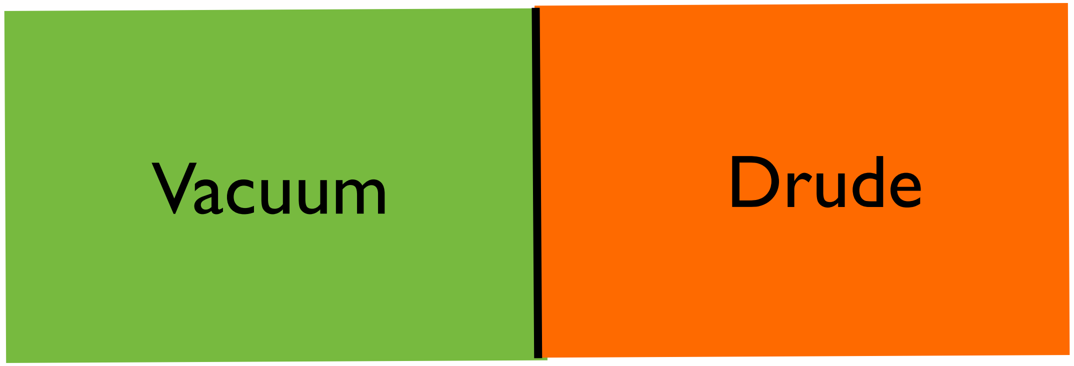

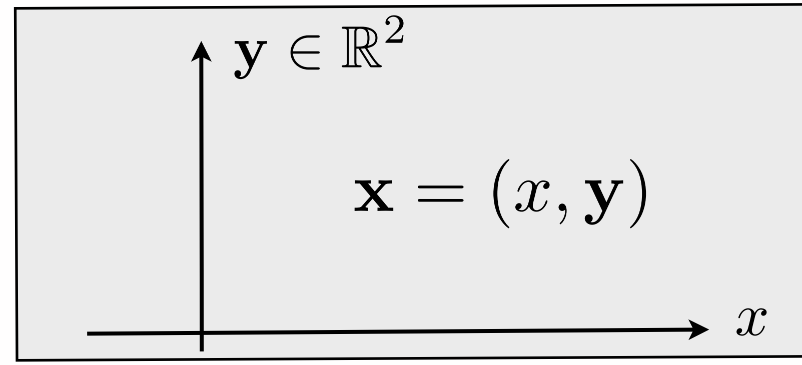

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Question : do we have for long time (limiting amplitude principle)

$$E(\mathbf{x}, t) \sim \mathbb{E}(\mathbf{x})\, e^{i\omega t} \qquad H(\mathbf{x}, t) \sim \mathbb{H}(\mathbf{x})\, e^{i\omega t}, \ldots \quad (t \to +\infty)$$

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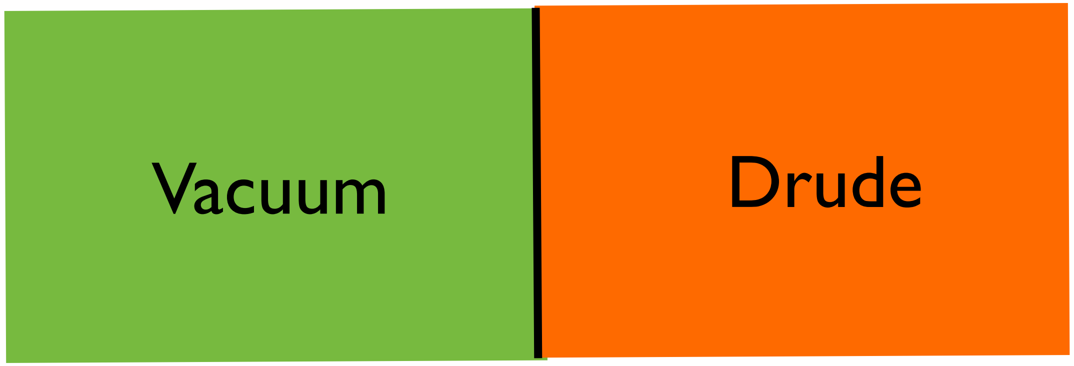

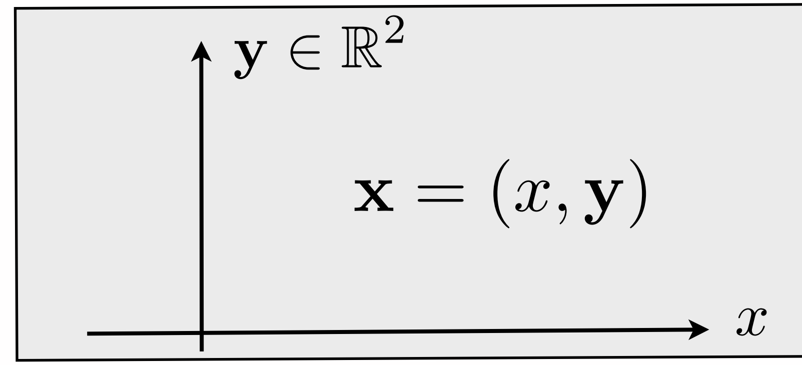

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Physical literature : Gralak-Tip (2010), Maystre-Gralak (2012)

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$\mathbf{x} = (x, \mathbf{y})$

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In our case, we expect that there in no limiting amplitude principle at frequencies for which the time harmonic problem is ill-posed

Bonnet-Ciarlet-Chesnel (2010-2012)

The transverse magnetic polarisation

$$\mathbf{y} = y \in \mathbb{R} \qquad (E, \boldsymbol{\Phi}) \in \mathbb{R} \times \mathbb{R} \qquad (\mathbf{H}, \boldsymbol{\Psi}) \in \mathbb{R}^2 \times \mathbb{R}^2$$

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Find : $\mathbf{U}(t) := \big(E(\cdot, t), \mathbf{H}(\cdot, t), \Phi(\cdot, t), \mathbf{\Psi}(\cdot, t) \big)^t \; : \mathbb{R}^+ \longrightarrow \mathcal{H}$

$$i \, \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \mathbb{A}\mathbf{U} = \mathbb{F}\, e^{i\omega t} \qquad \mathbf{U}(0) = 0$$

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$\mathbb{A}\mathbf{U} := \mathcal{A}\,\mathbf{U}$ with domain $D(\mathbb{A})$ (omitted here)

$$\mathcal{A} := -\,i \begin{pmatrix} 0 & \varepsilon_0^{-1}\,\mathrm{rot} & -\Omega_e^2\,\Pi & 0 \\ \mu_0^{-1}\,\mathbf{rot} & 0 & 0 & -\Omega_m^2\,\Pi \\ \chi & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 \end{pmatrix}$$

$$\mathrm{rot}\, u := (\partial_y u, -\partial_x u)$$

$$\mathbf{rot}\, \mathbf{u} := \partial_x \mathbf{u}_y - \partial_y \mathbf{u}_x$$

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Lemma : $\mathbb{A}$ is self-adjoint in $\mathcal{H}$ equipped with an adequate weighted L^2 inner product denoted

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Corollary : The evolution problem is well-posed and

$$\mathbf{U}(t) = \int_0^t e^{-i(t-s)\,\mathbb{A}} \, \mathbb{F} \, e^{i\omega s} \, ds \qquad \|\mathbf{U}(t)\|_{\mathcal{H}} \leq t \, \|\mathbb{F}\|_{\mathcal{H}}$$

The main results

The non critical case : $\Omega_e \neq \Omega_m$

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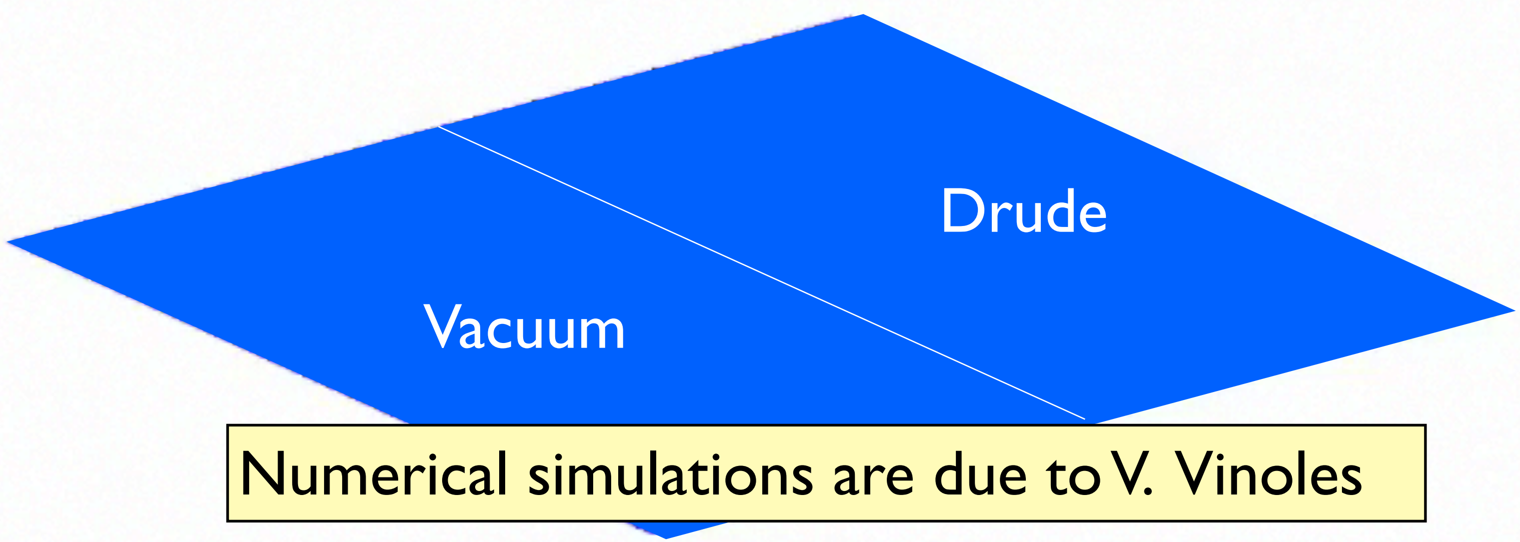

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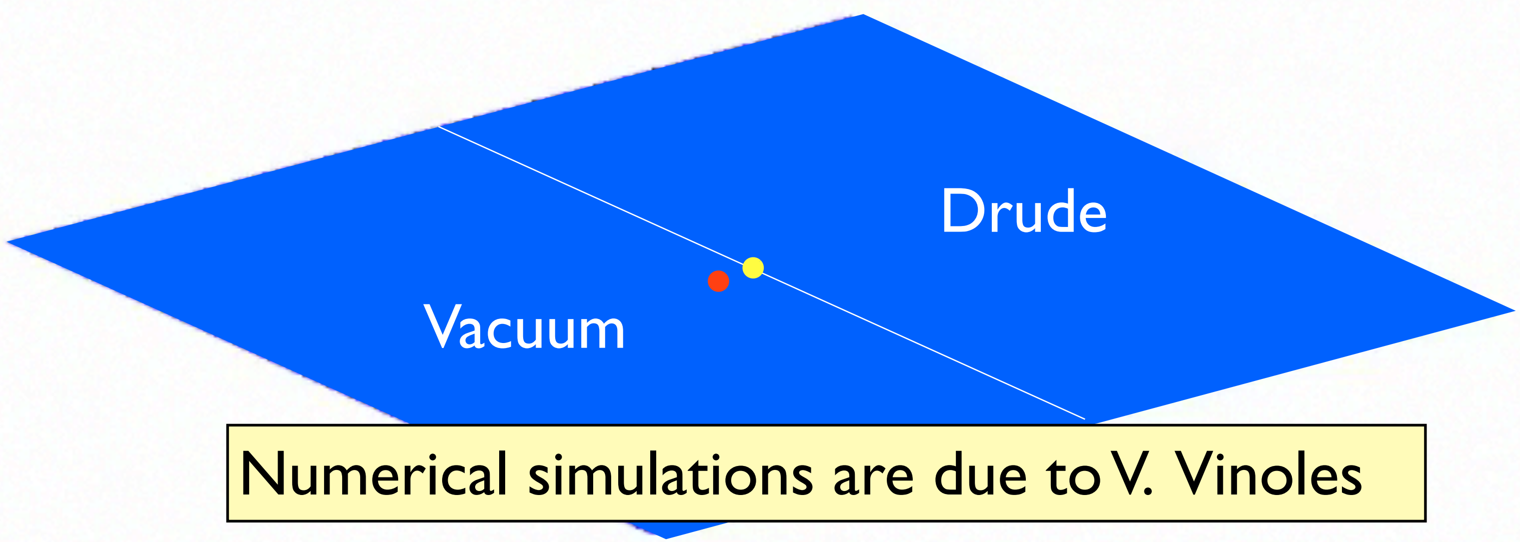

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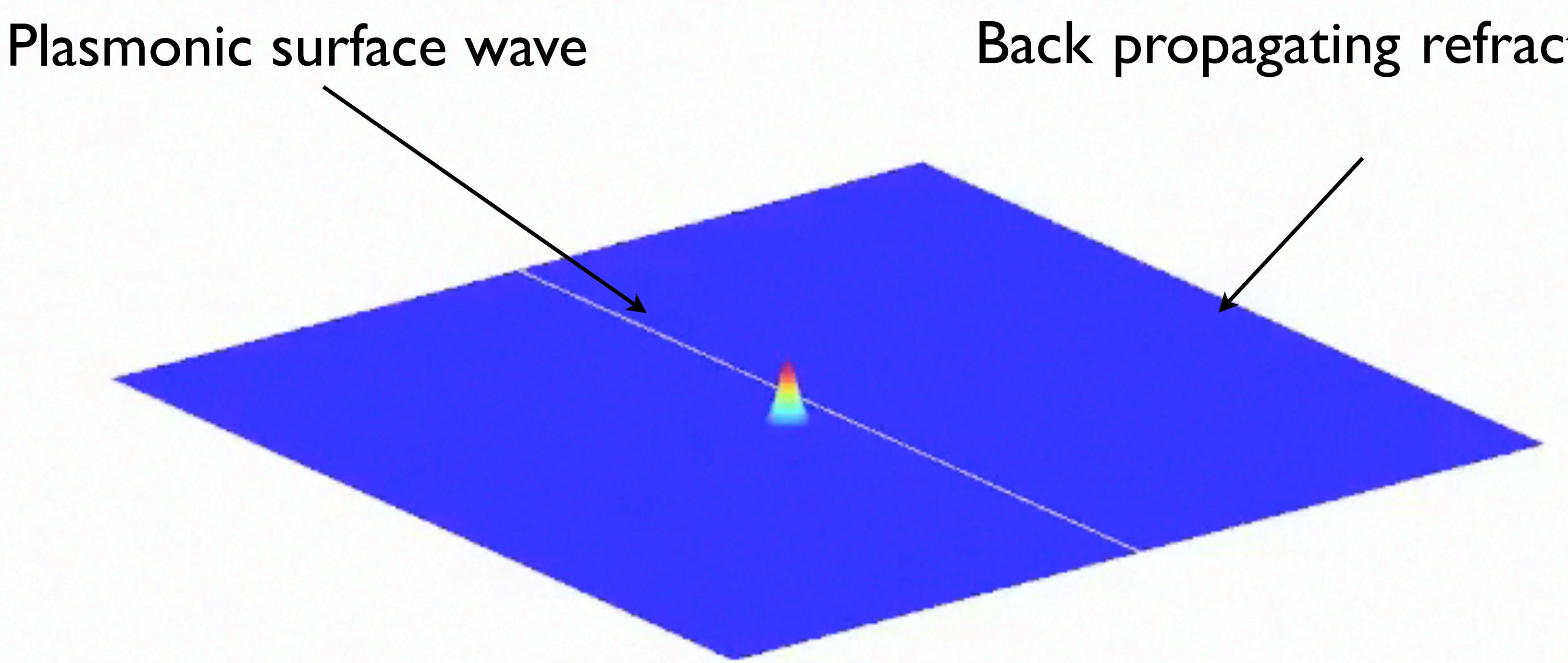

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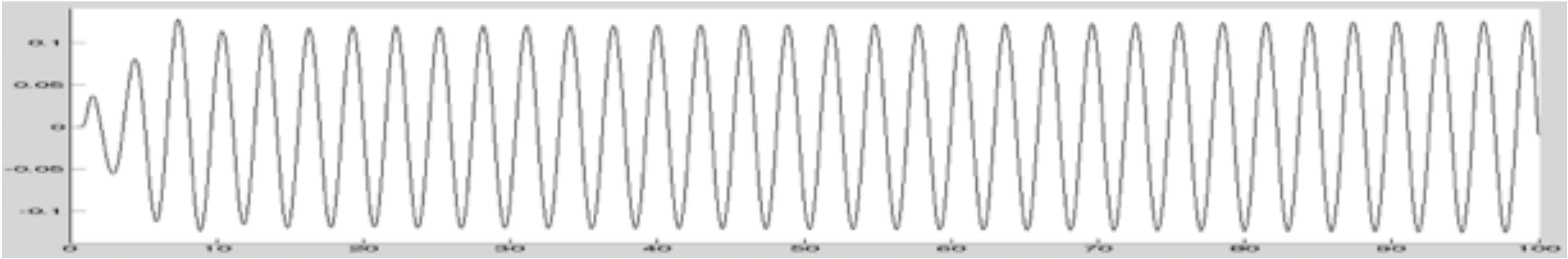
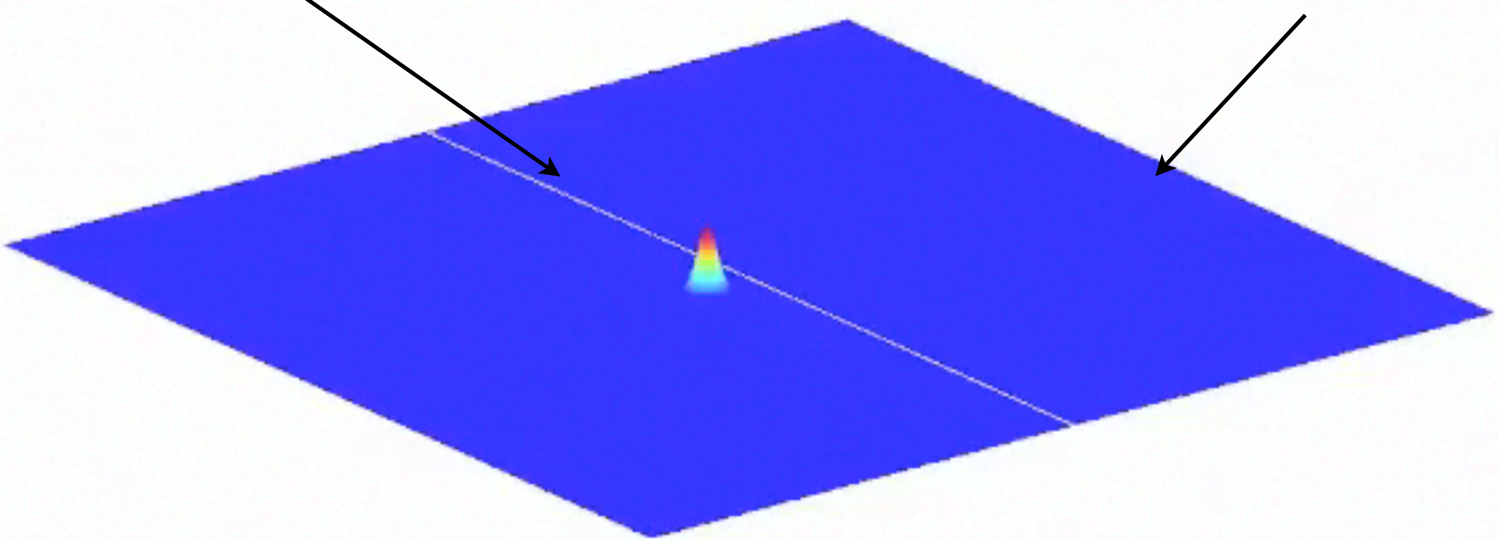


Plasmonic surface wave

Back propagating refracted wave



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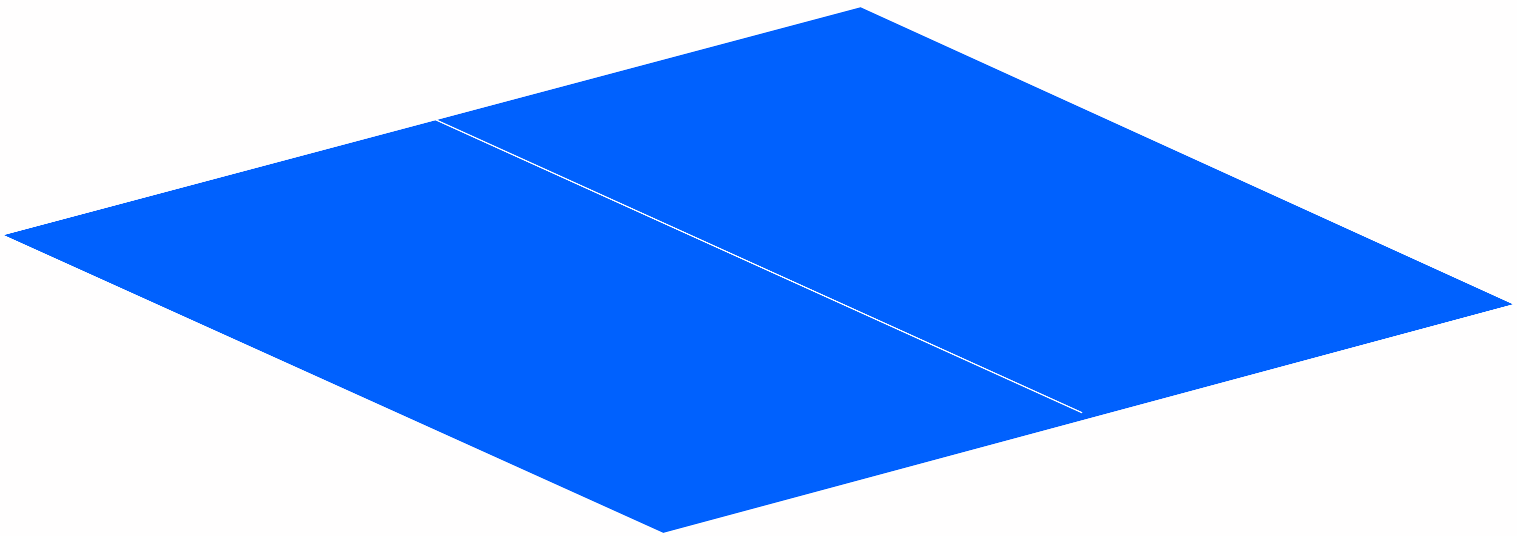

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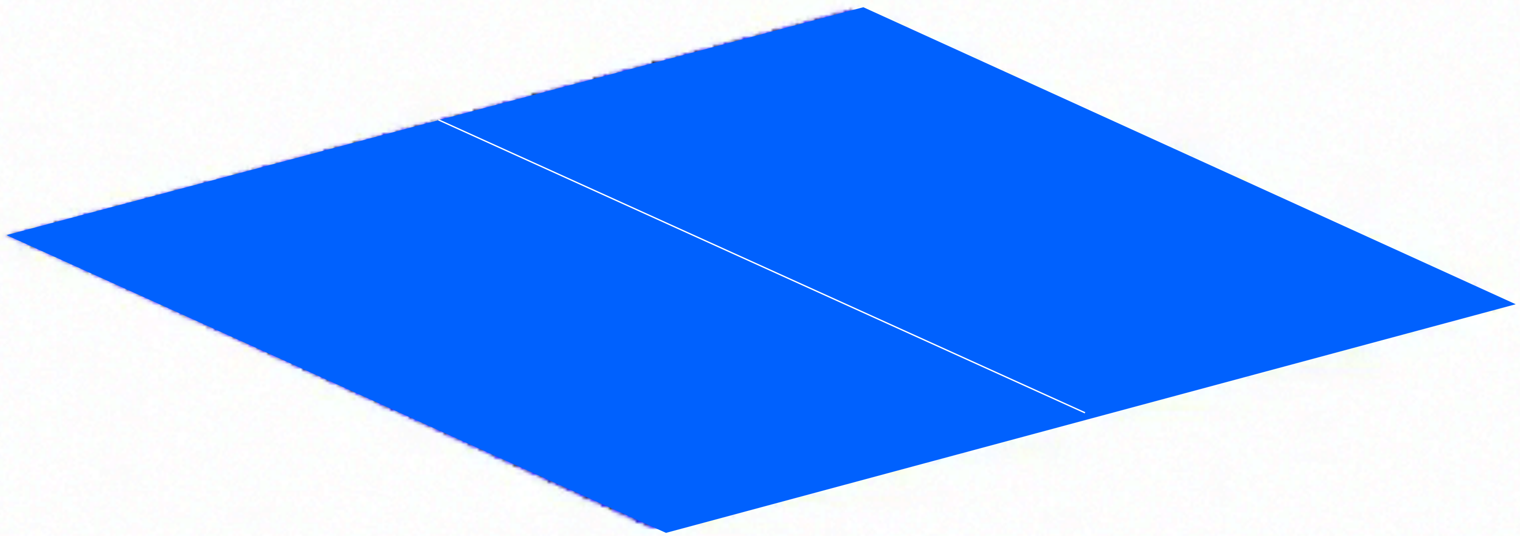

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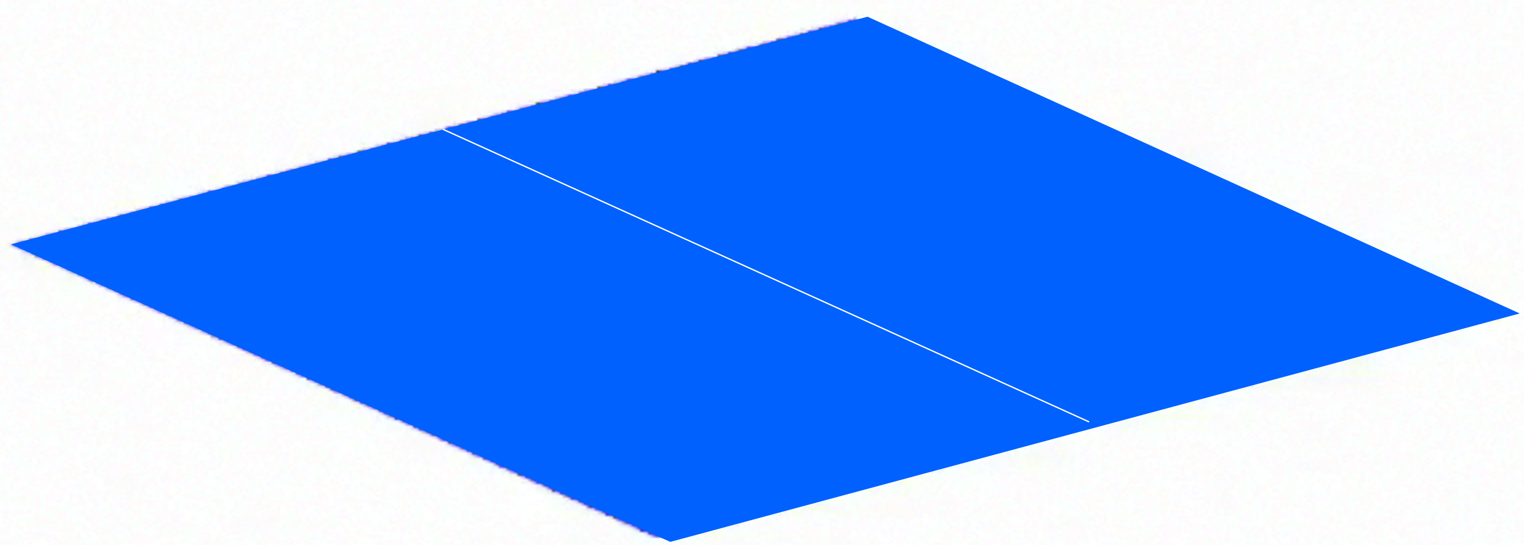

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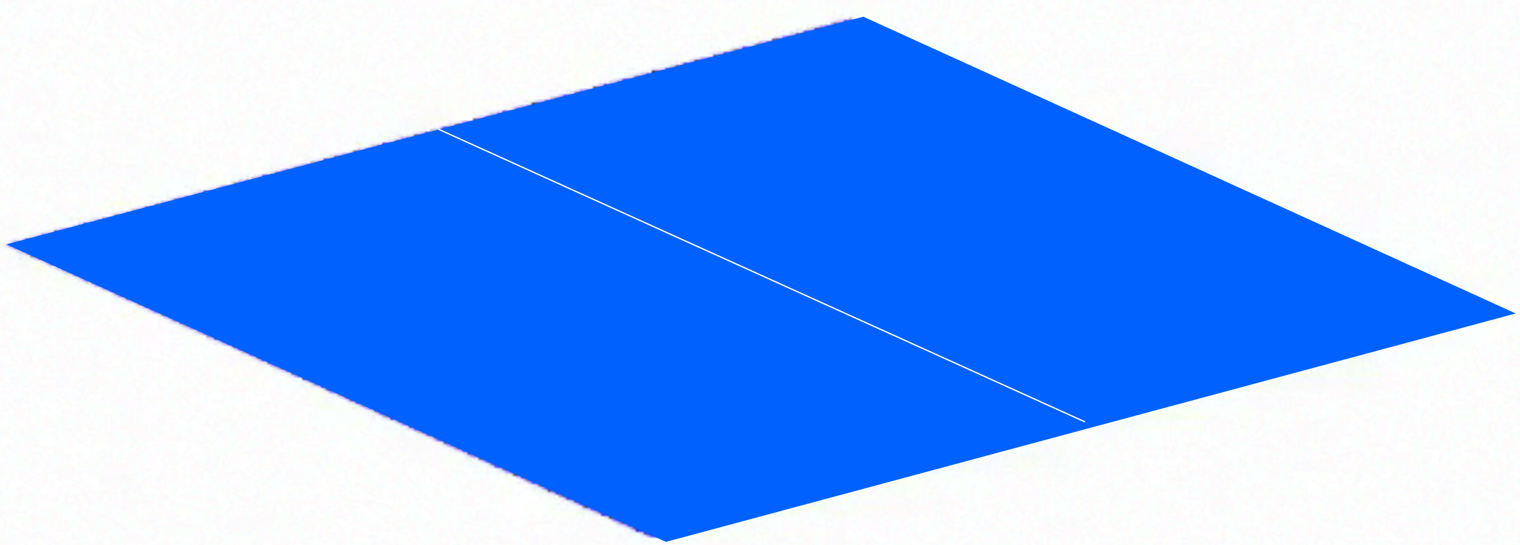
$$E(x,t) = \mathbb{E}(x)\, e^{i\omega t} + \sum_{\pm} \mathbb{E}_p^{\pm}(x)\, e^{\pm\, i\omega_p t} + o(1) \qquad (t \to +\infty)$$



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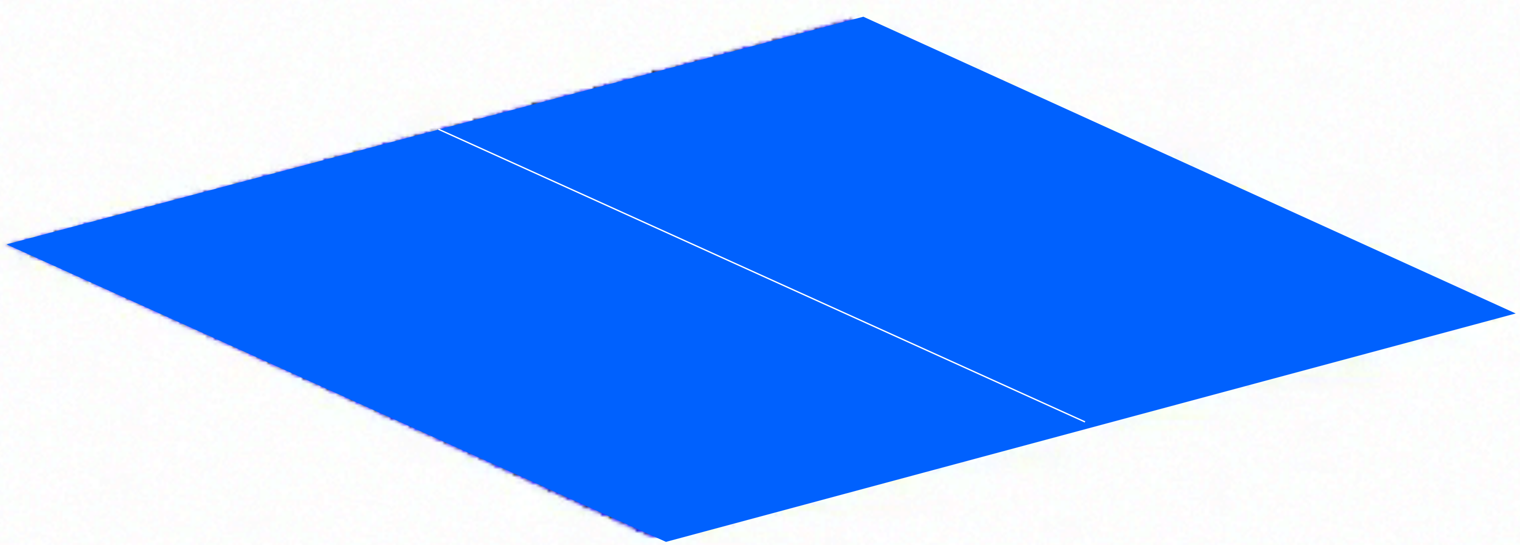

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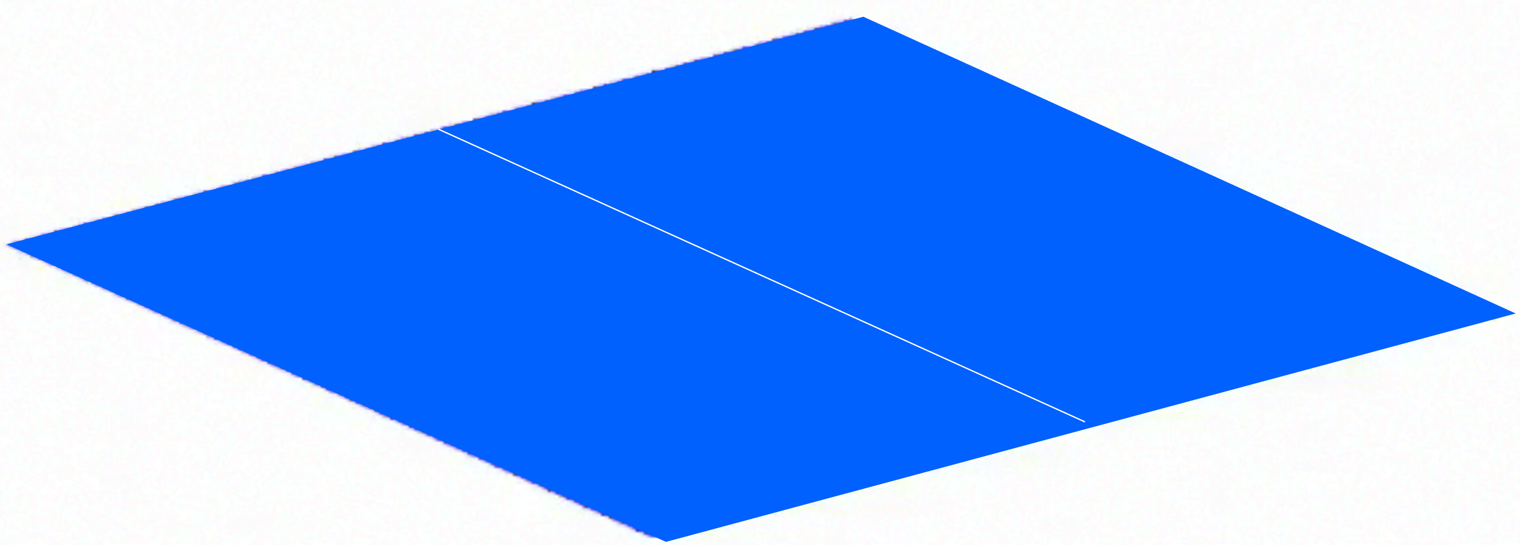

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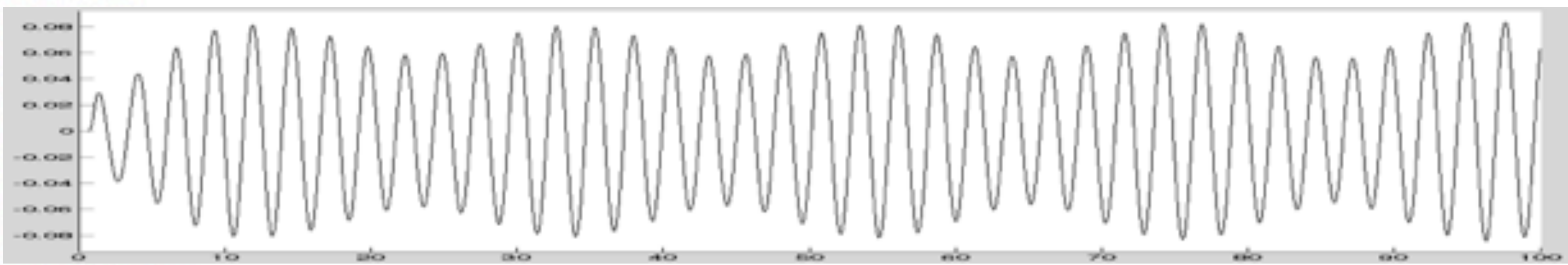
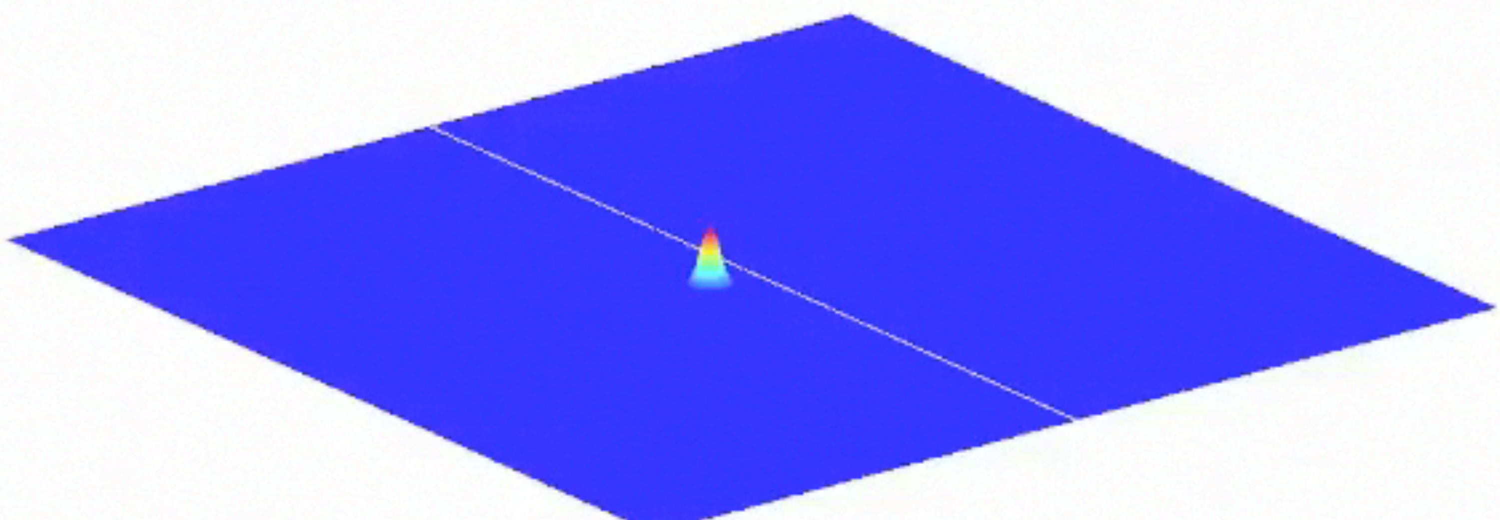


Beat phenomenon



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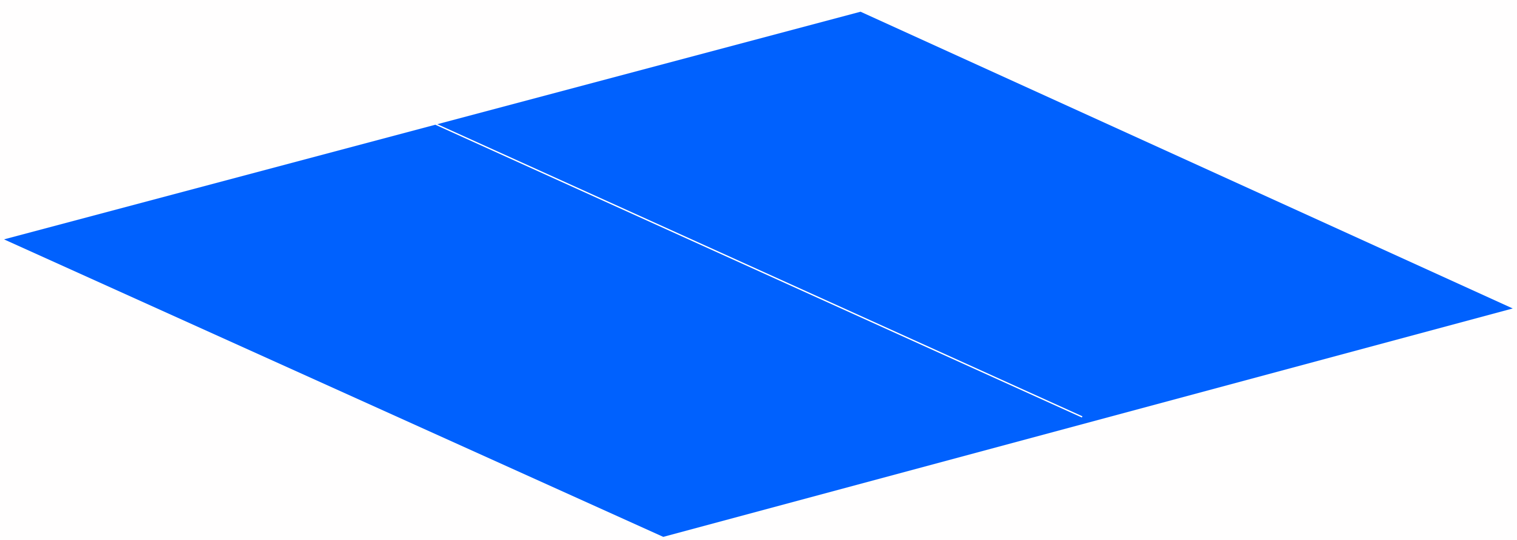

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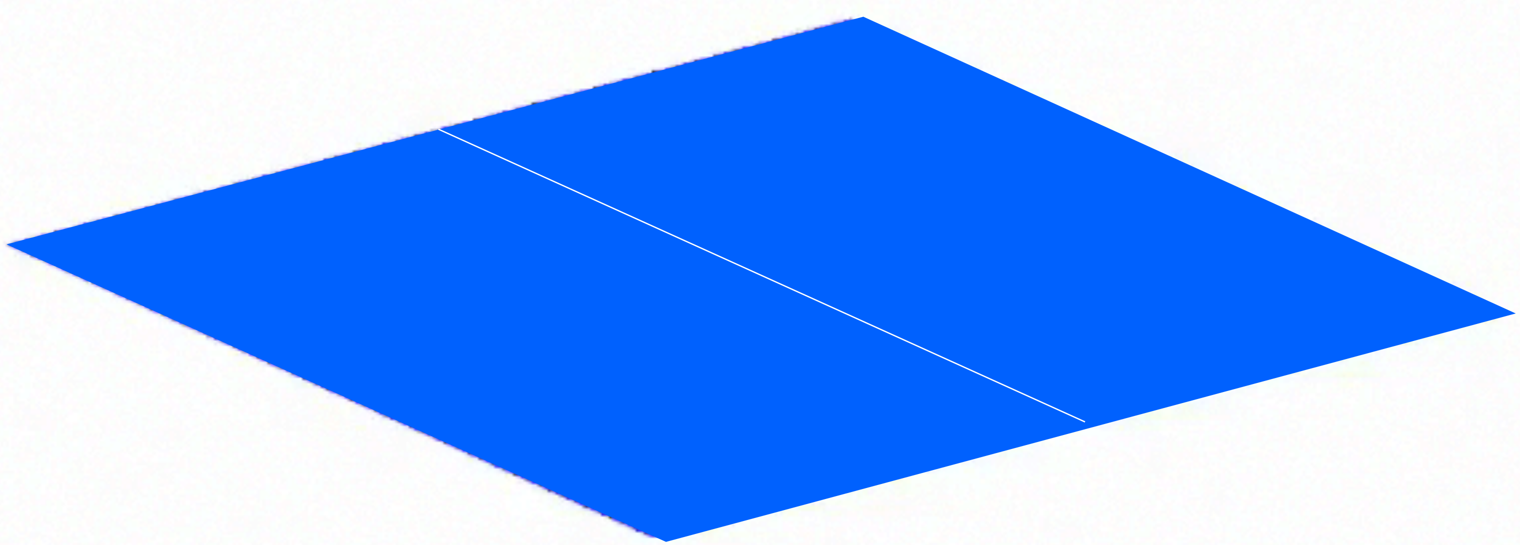

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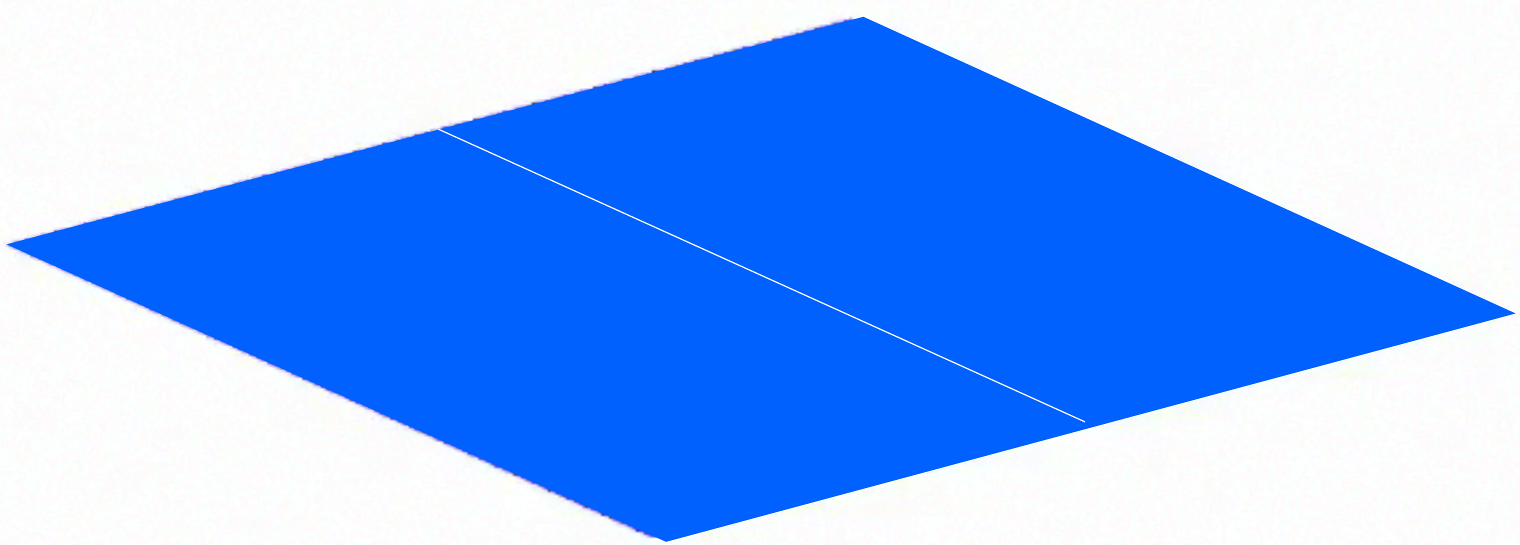

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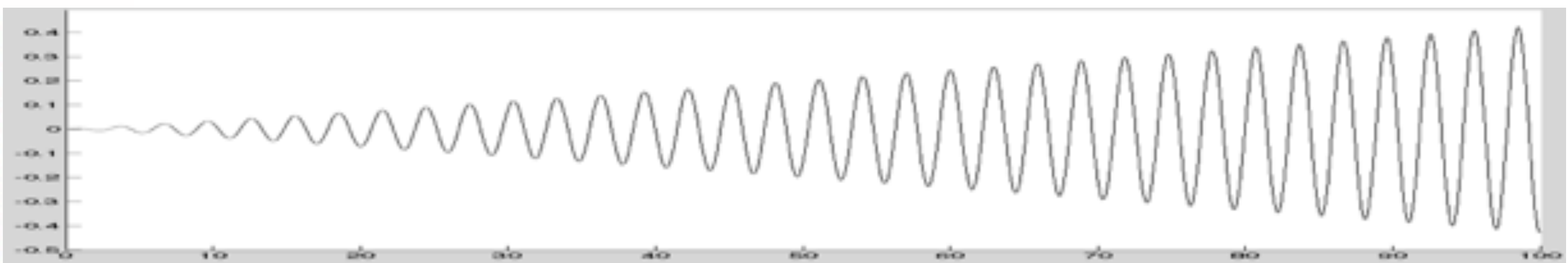
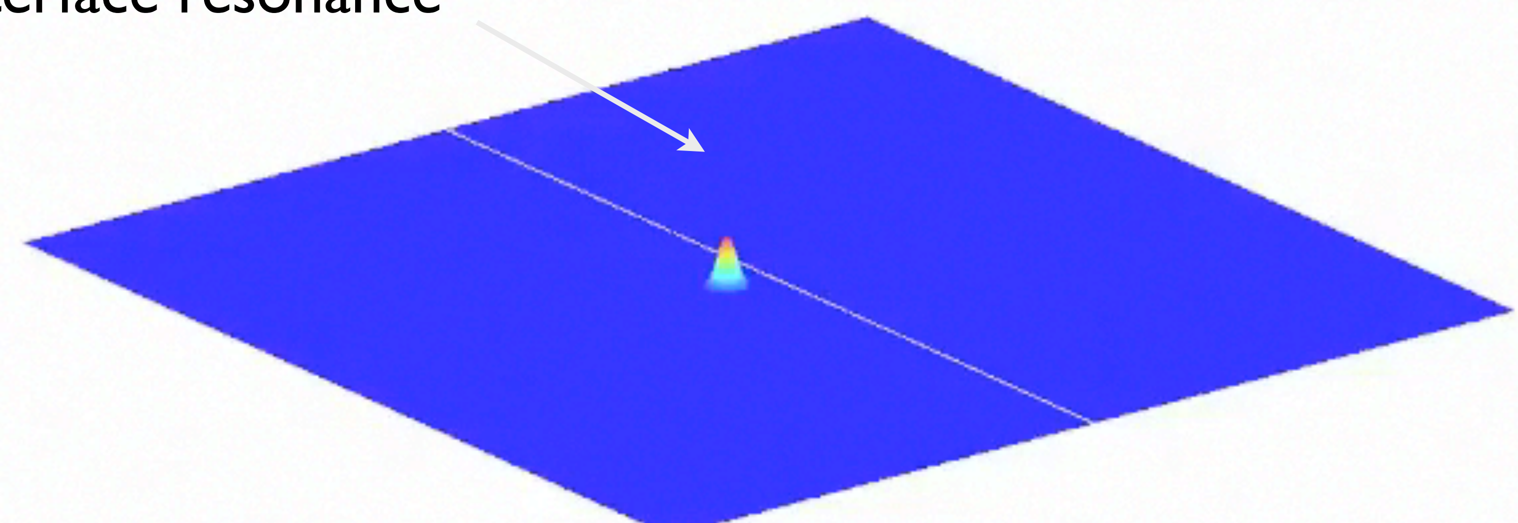


Interface resonance

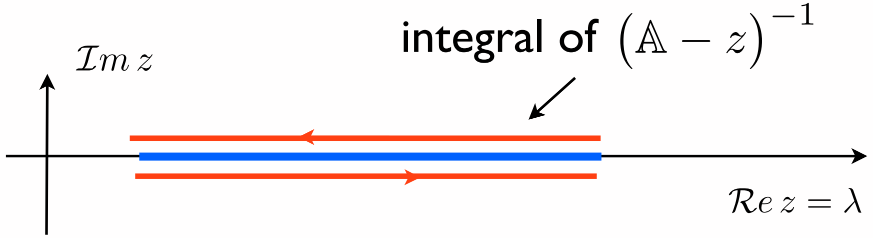


The method of analysis

The method of analysis

Spectral theory : spectral projectors

Stone's formula for
spectral projectors



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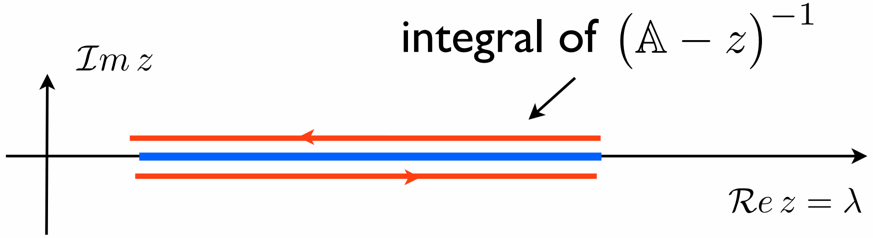

Uses an integral representation of the resolvent of $\mathbb{A}$ by construction of the adequate Green's function

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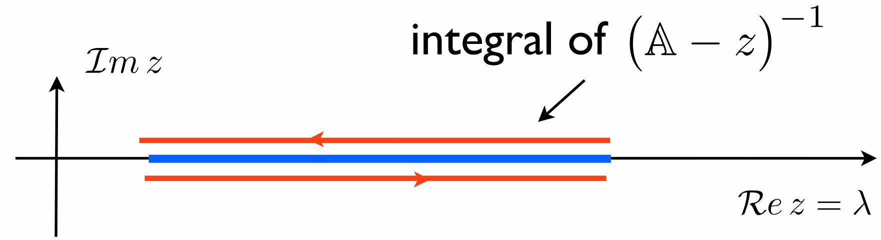

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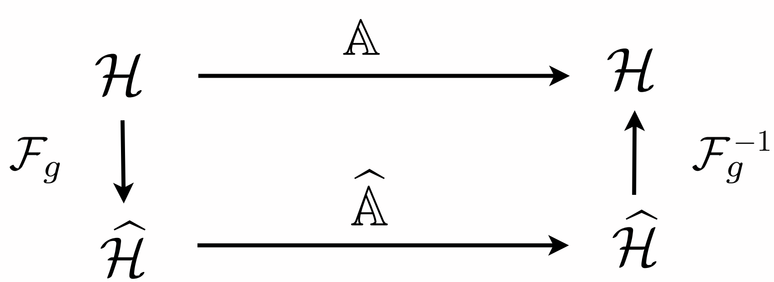
Construction of a generalized Fourier transform



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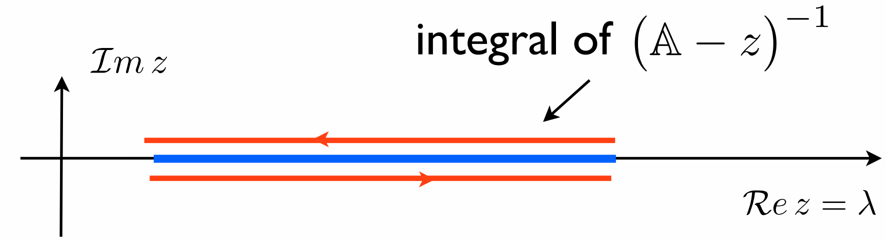

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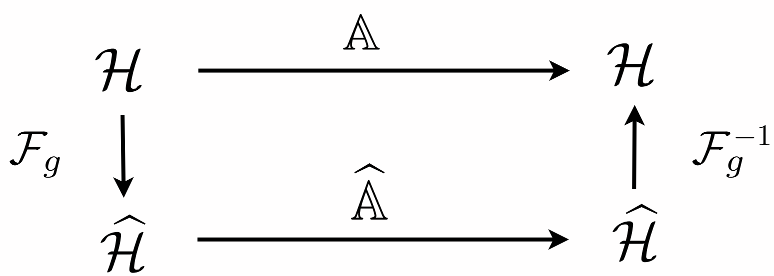

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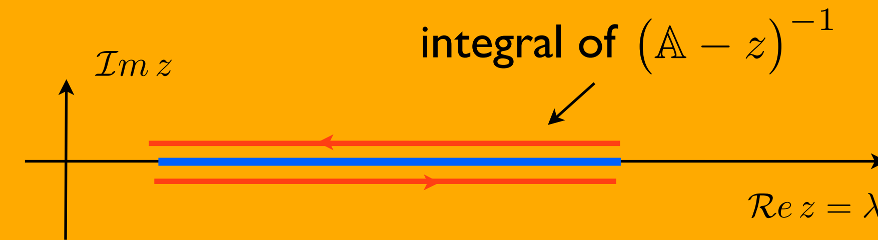

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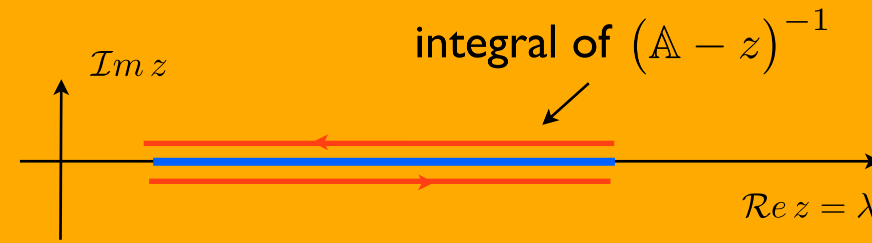

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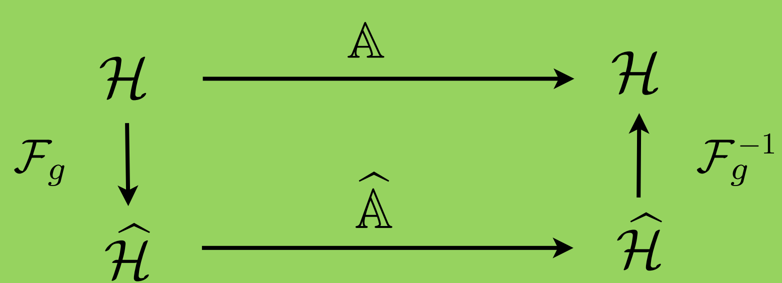

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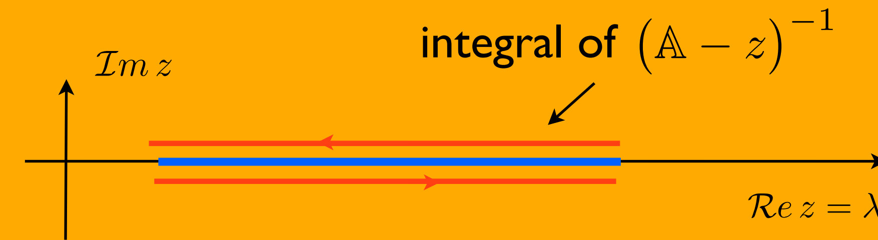

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Modal analysis

Particular solutions of the form $\big((\lambda, k) \in \mathbb{R}^2, \text{given}\big)$

$$\mathbf{U}(x,y,t) := e^{i\lambda t}\, e^{iky}\, \mathbf{W}(x) \qquad\qquad |\mathbf{W}(x)| \leq C\,\big(1 + |x|\,\big)$$

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If $\mathbf{W} = \big(E, \mathbf{H}, \Phi, \mathbf{\Psi}\big)^T$ one can eliminate $\big(\mathbf{H}, \Phi, \mathbf{\Psi}\big)$ to get

$$-\mu(\lambda,x)\frac{d}{dx}\Big(\mu(\lambda,x)^{-1}\frac{dE}{dx}\Big) + \big(\,k^2 - \lambda^2\,\varepsilon(\lambda,x)\mu(\lambda,x)\,\big)\,E = 0$$

$$\mu(\lambda,x) = \mu_0 \text{ for } x<0, \qquad \mu(\lambda,x) = \mu(\lambda) \text{ for } x>0 \qquad \mu(\lambda) := \mu_0\Big(1-\frac{\Omega_m^2}{\lambda^2}\Big)$$

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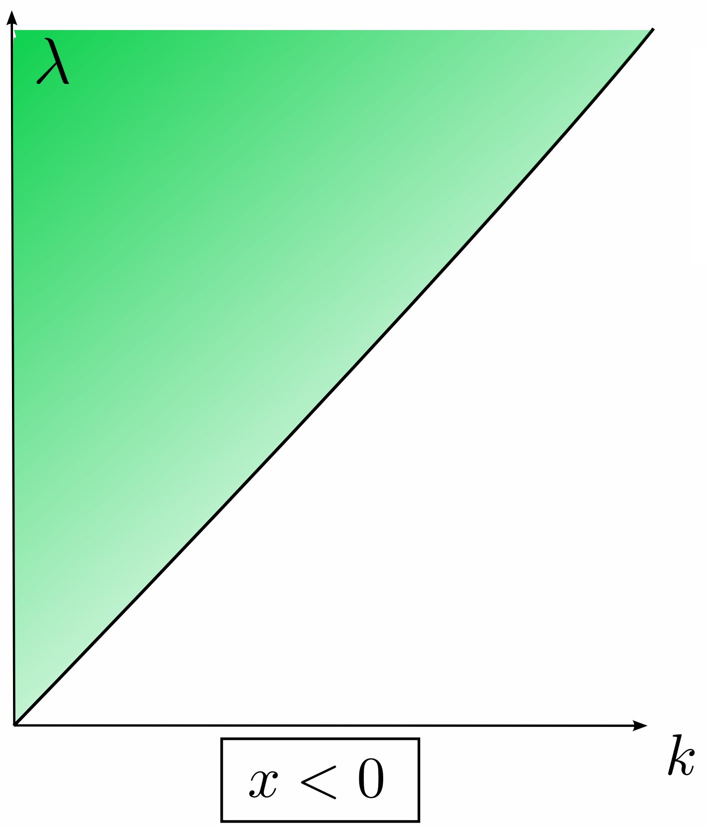

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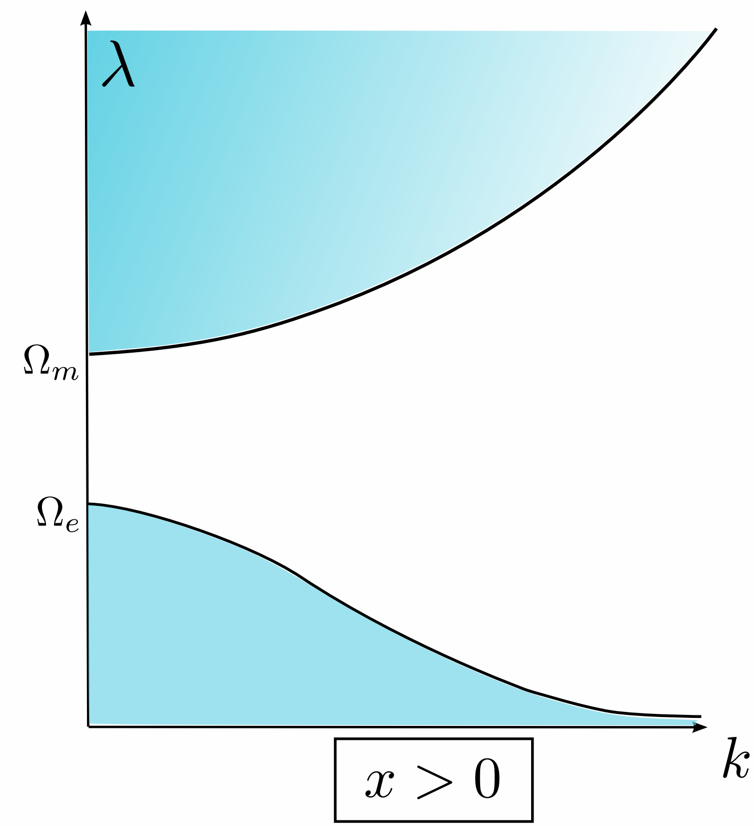

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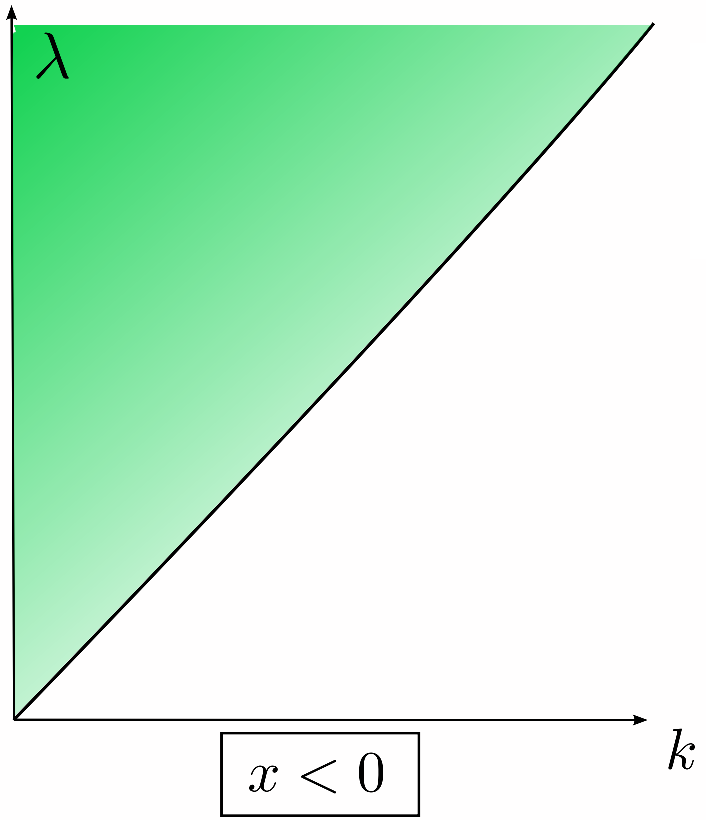

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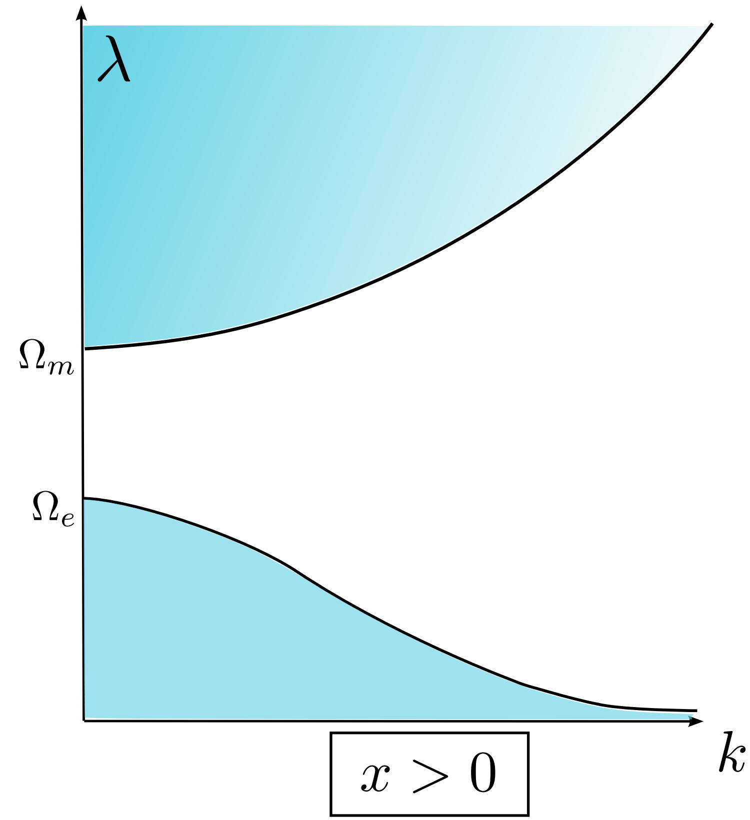

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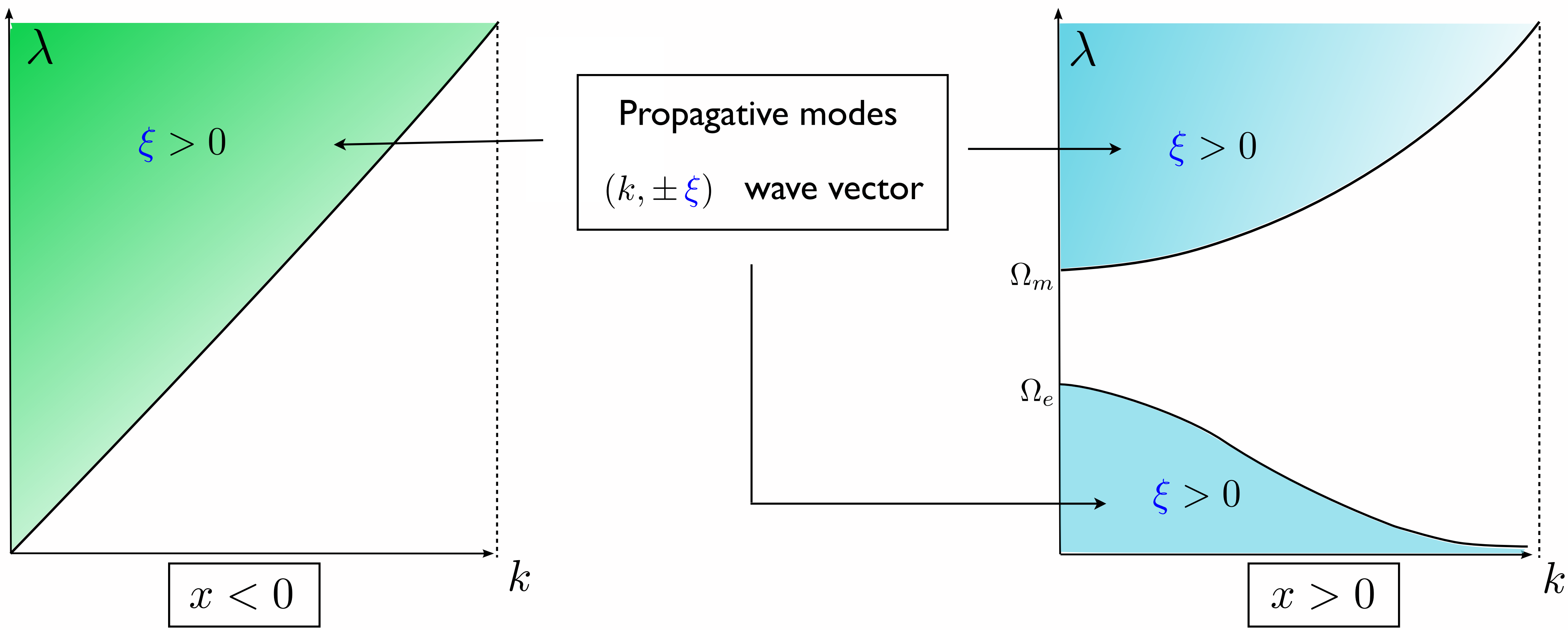

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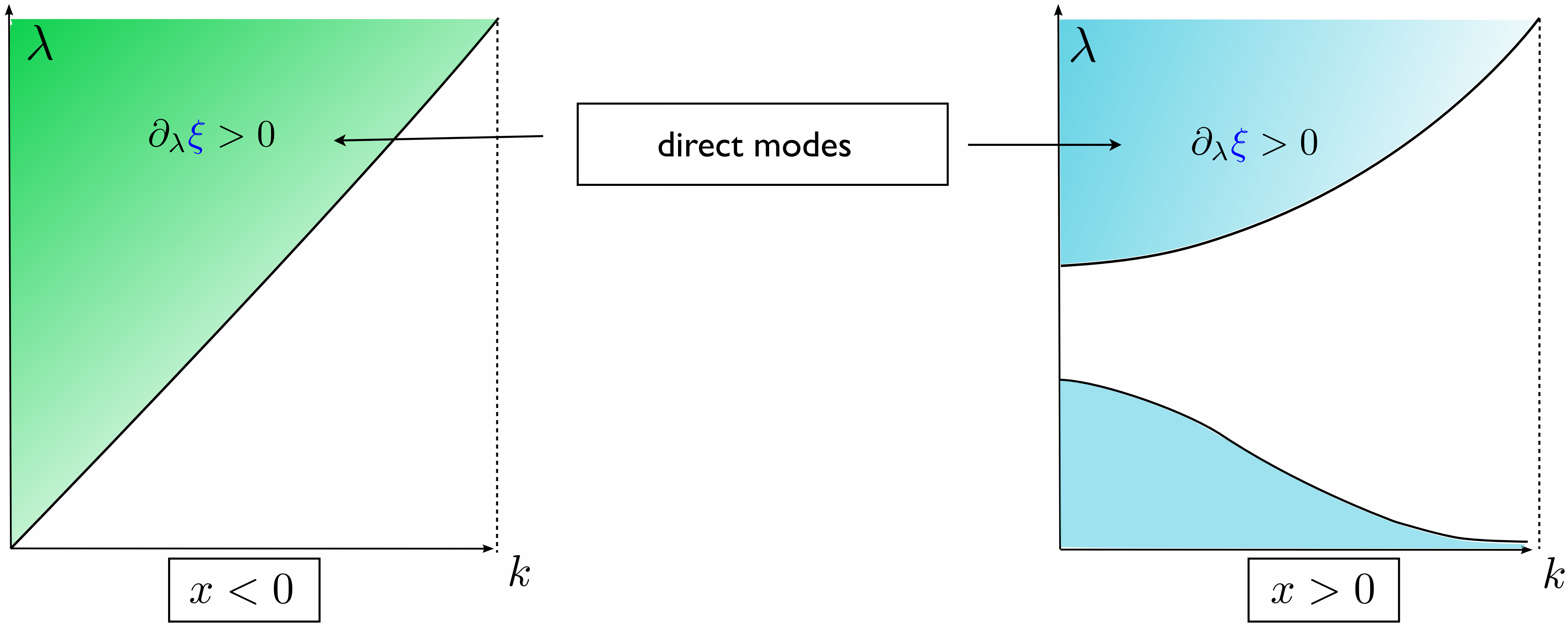

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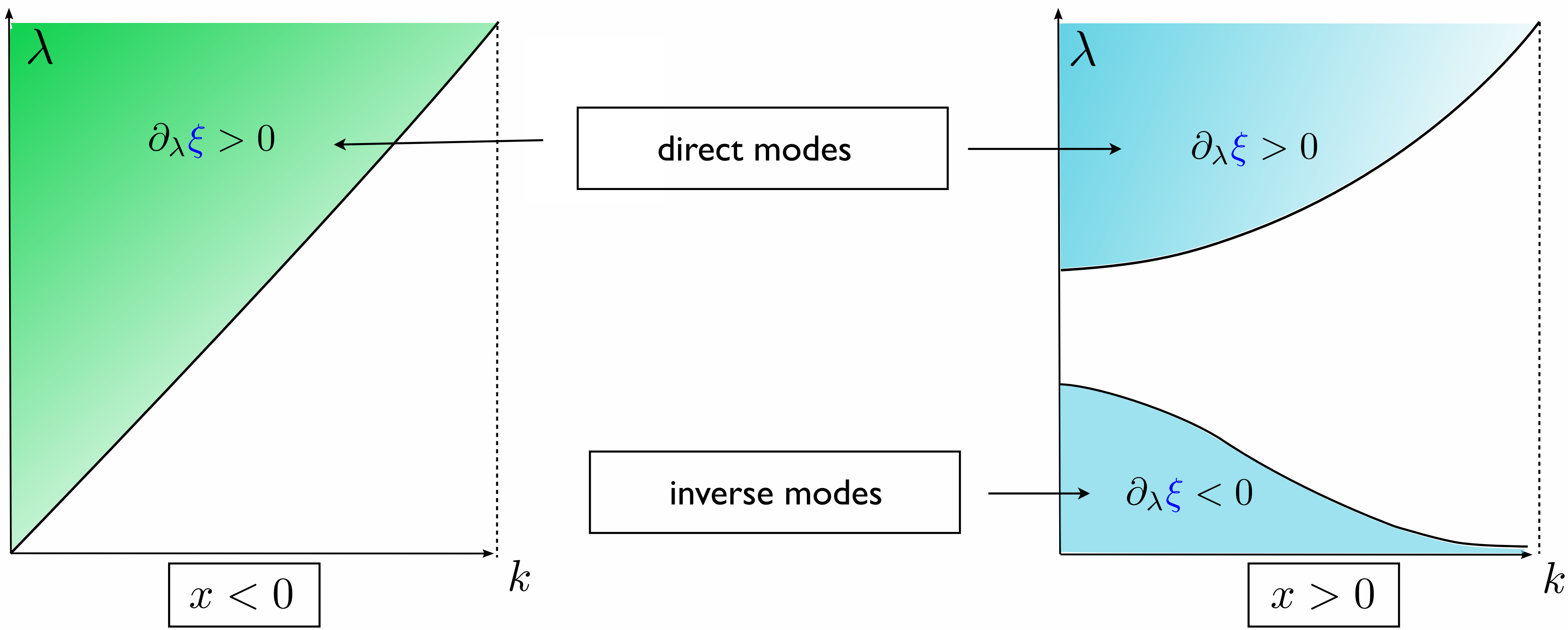

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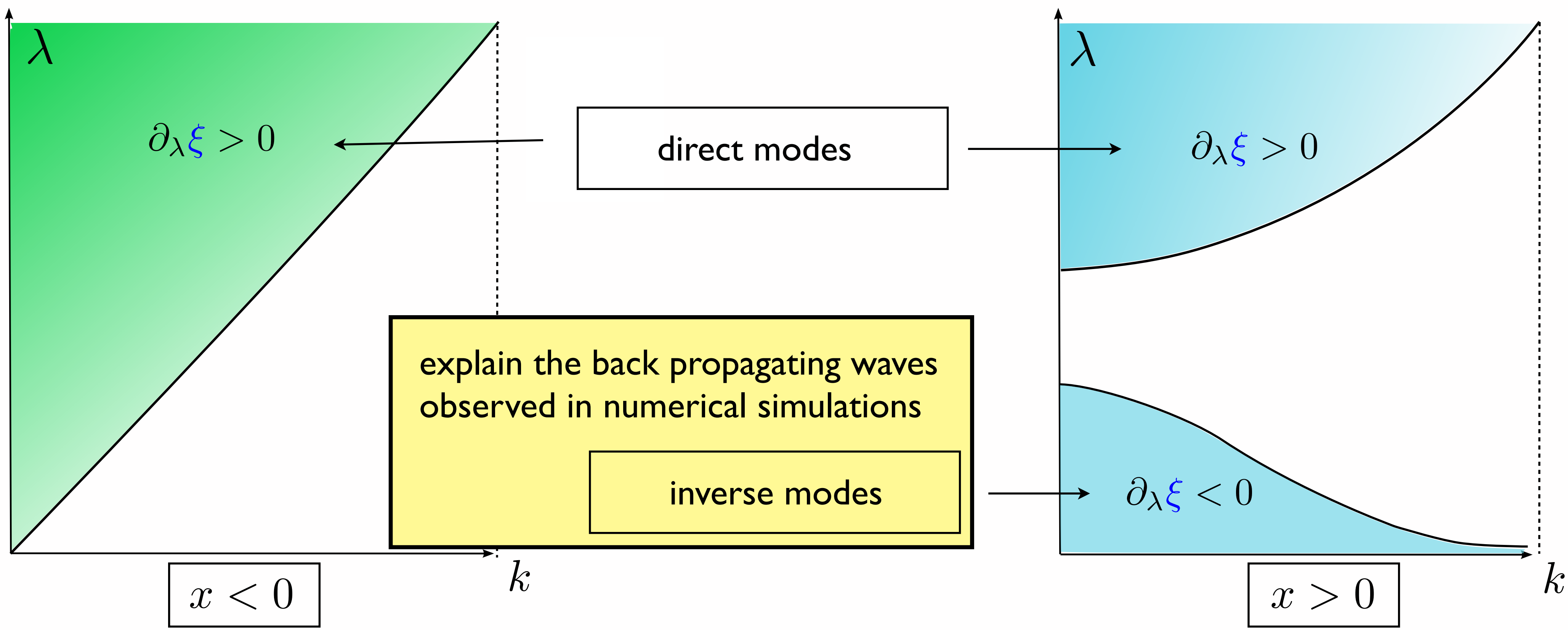

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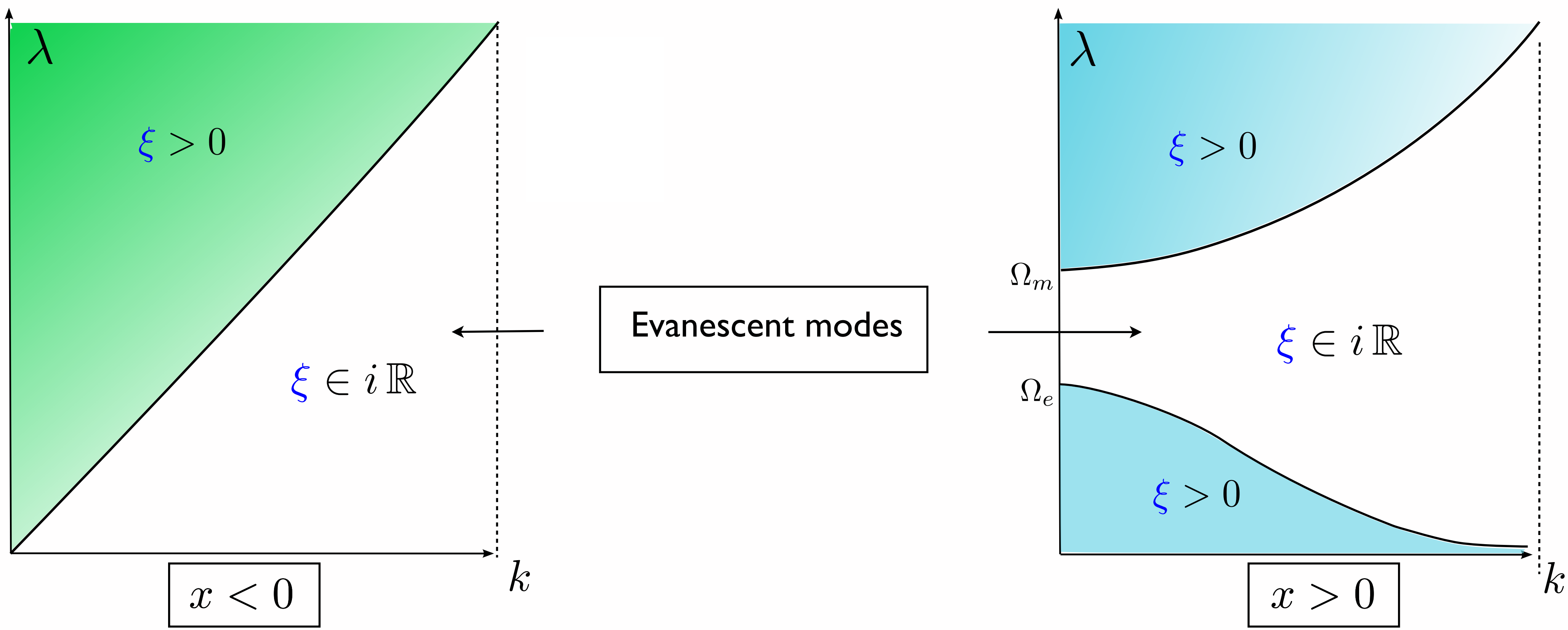

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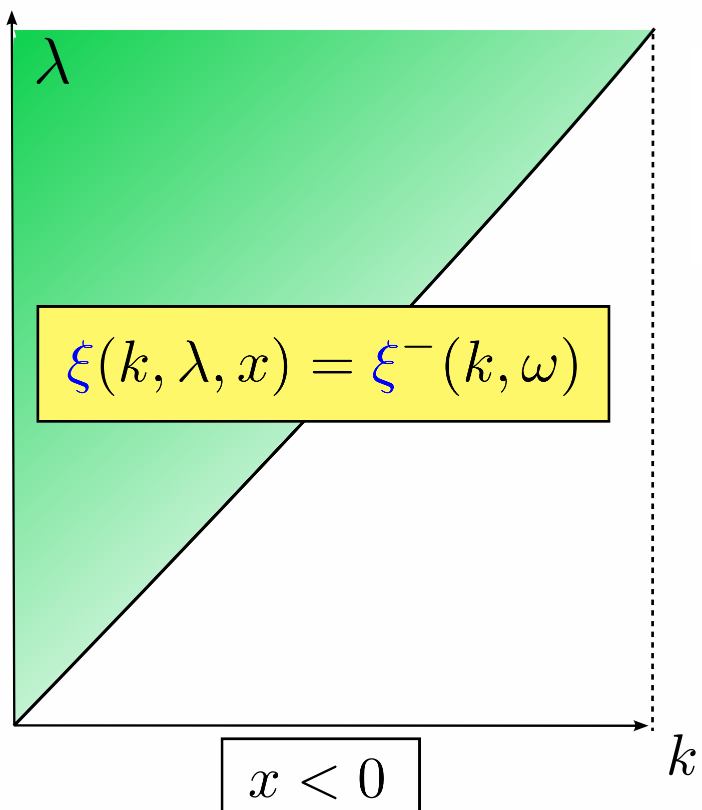
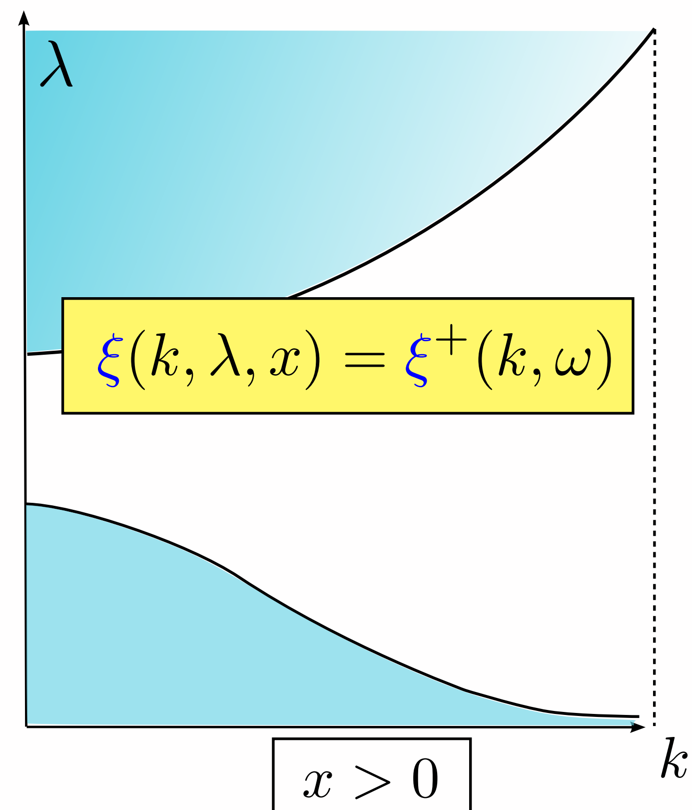


It remains to satisfy two transmission conditions at $x=0$



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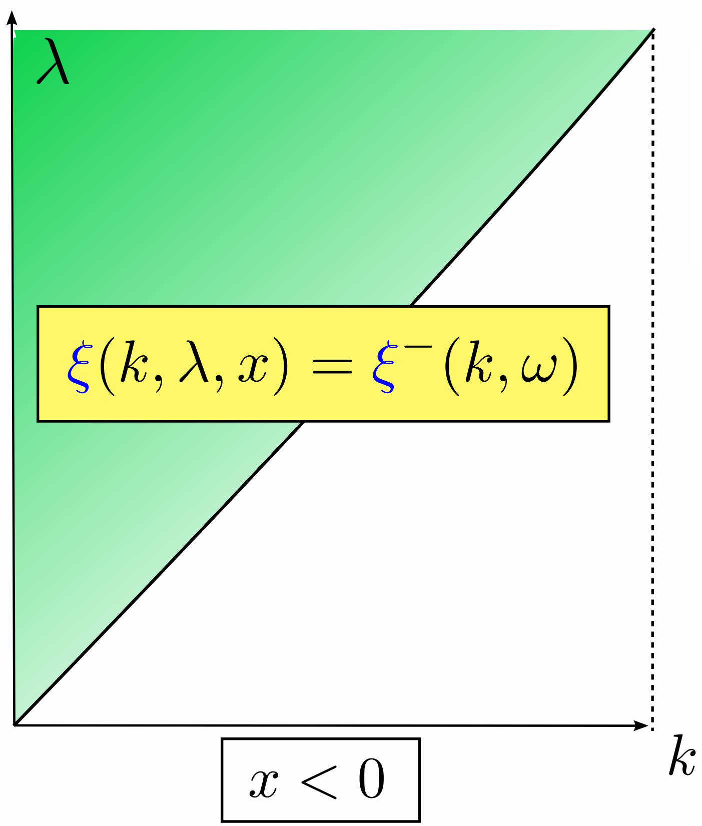

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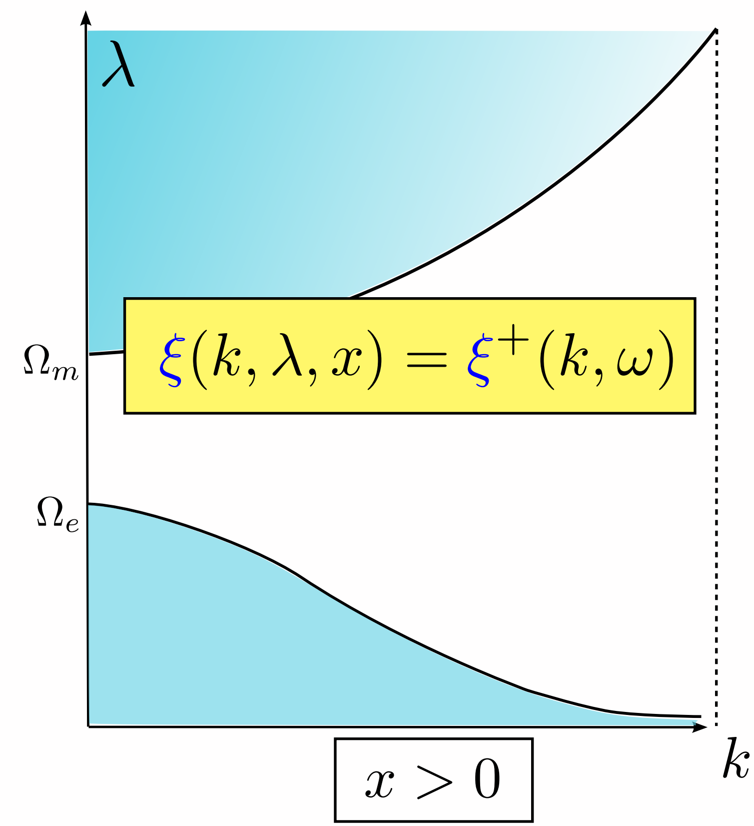

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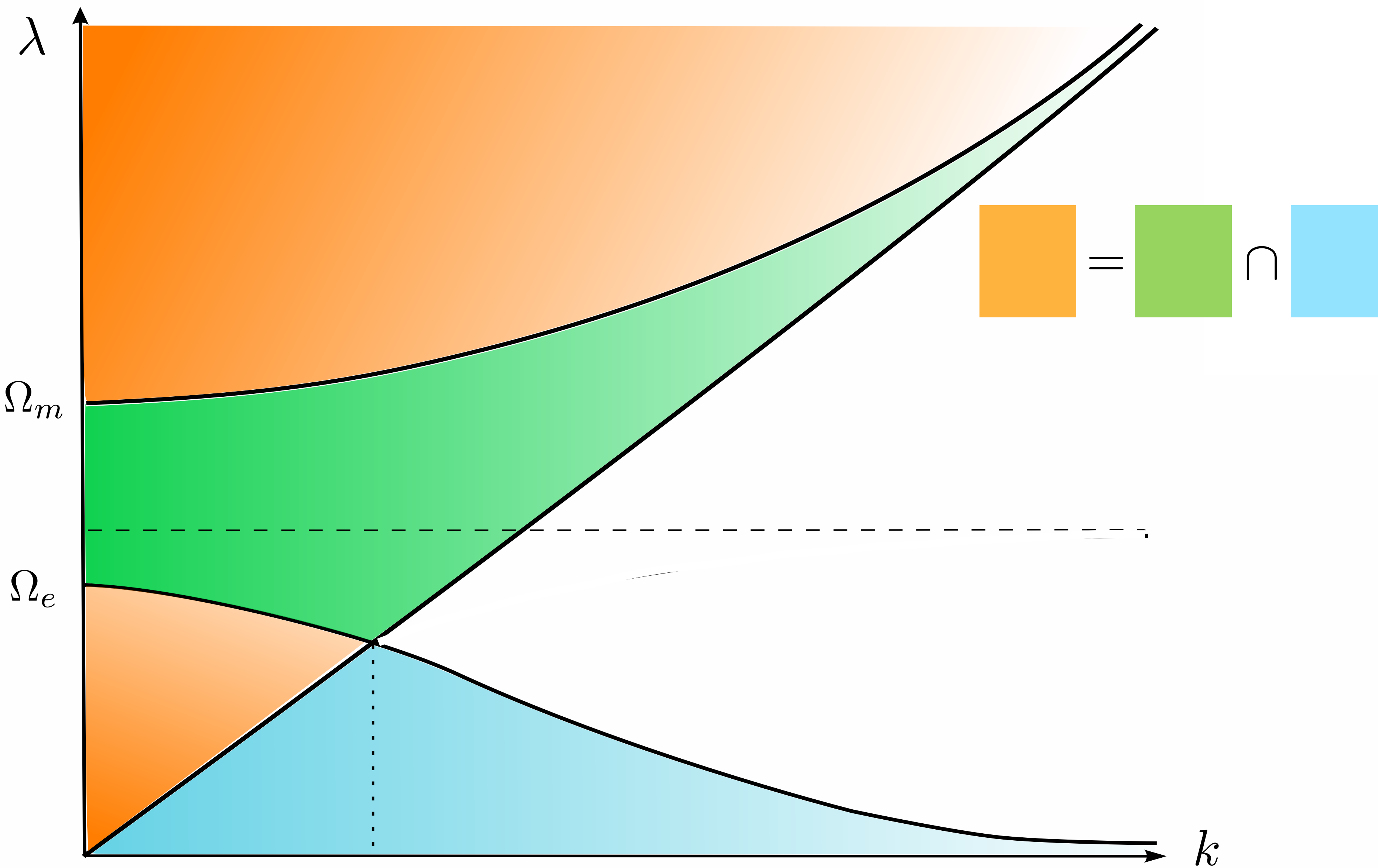

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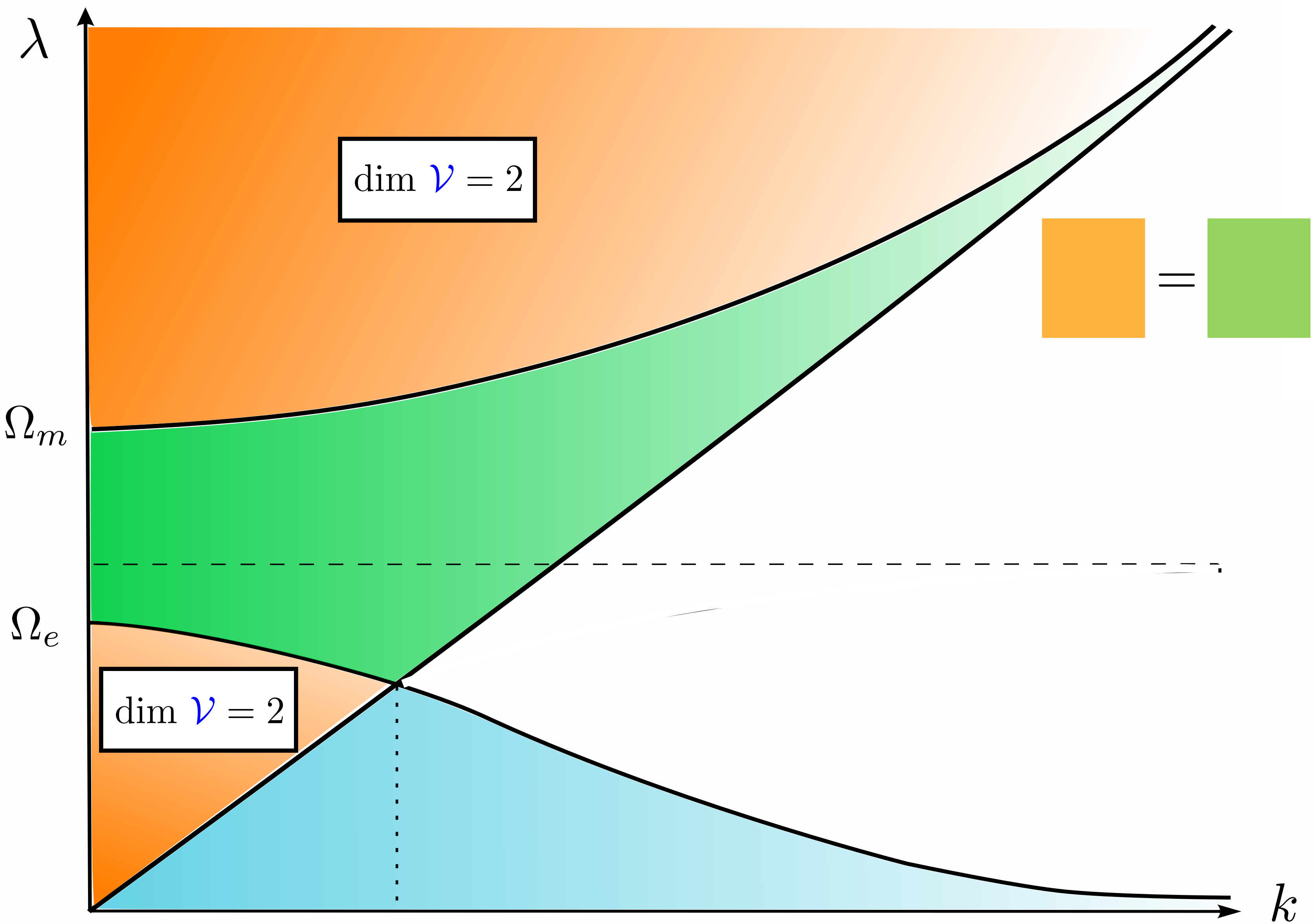

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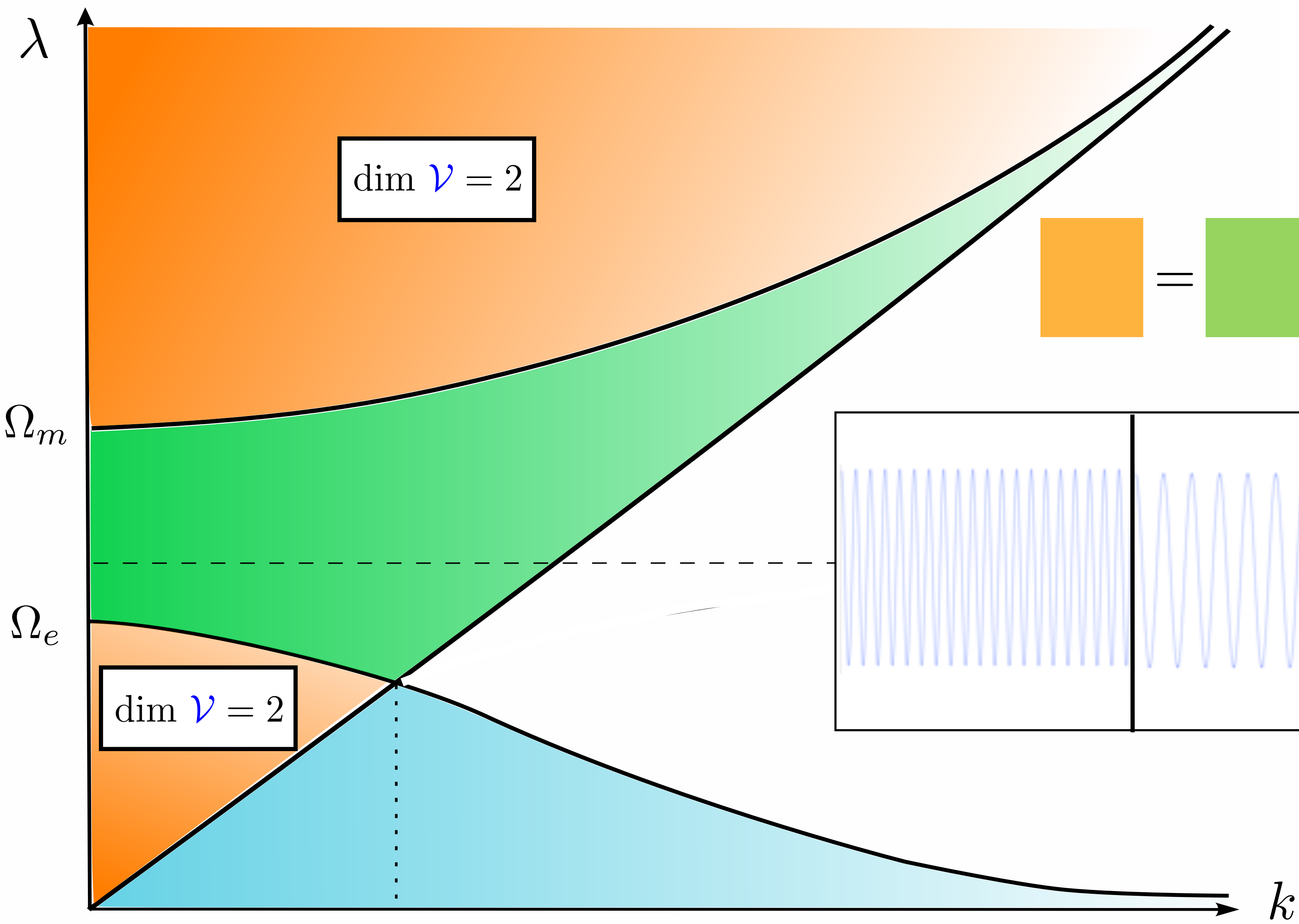

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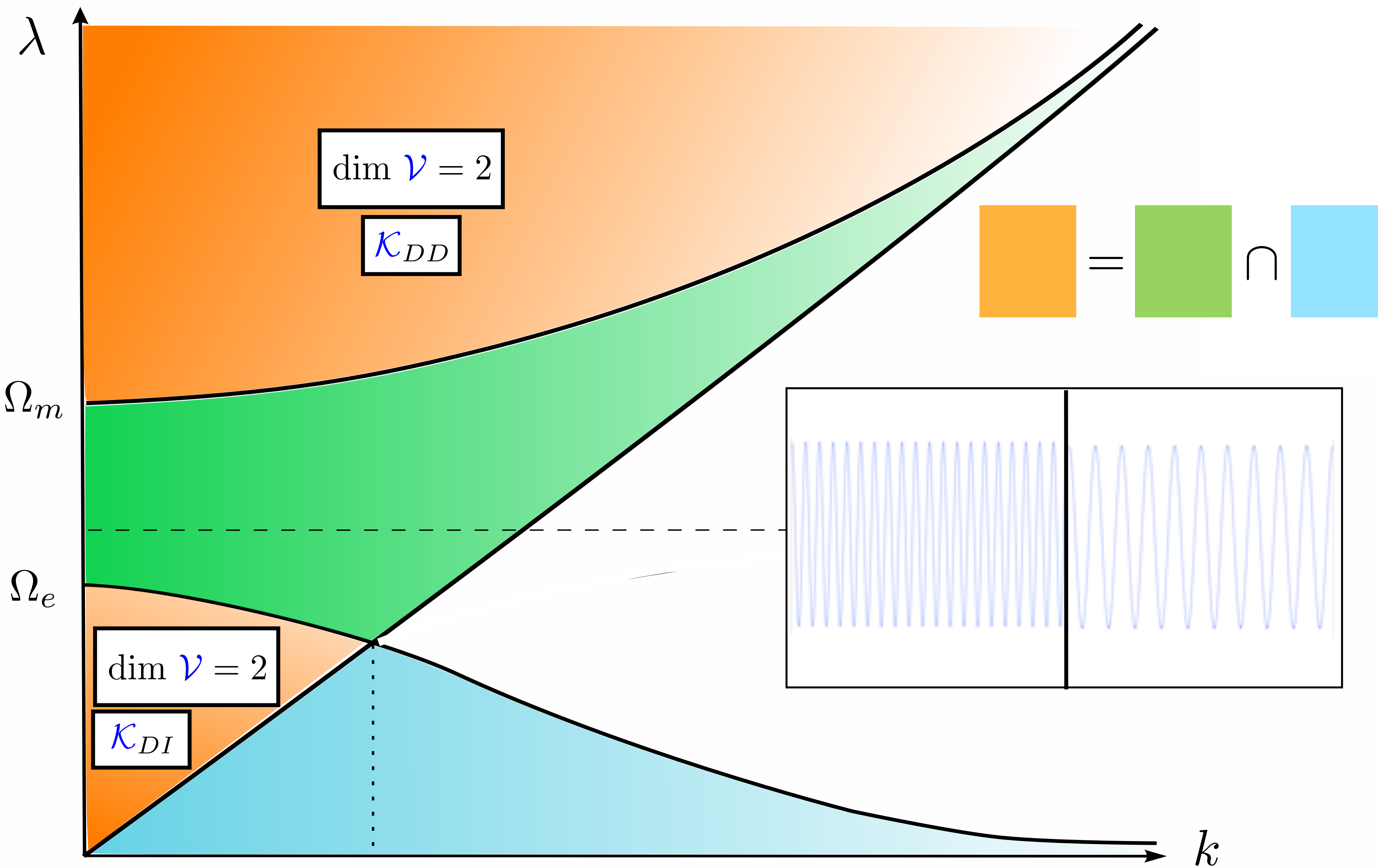

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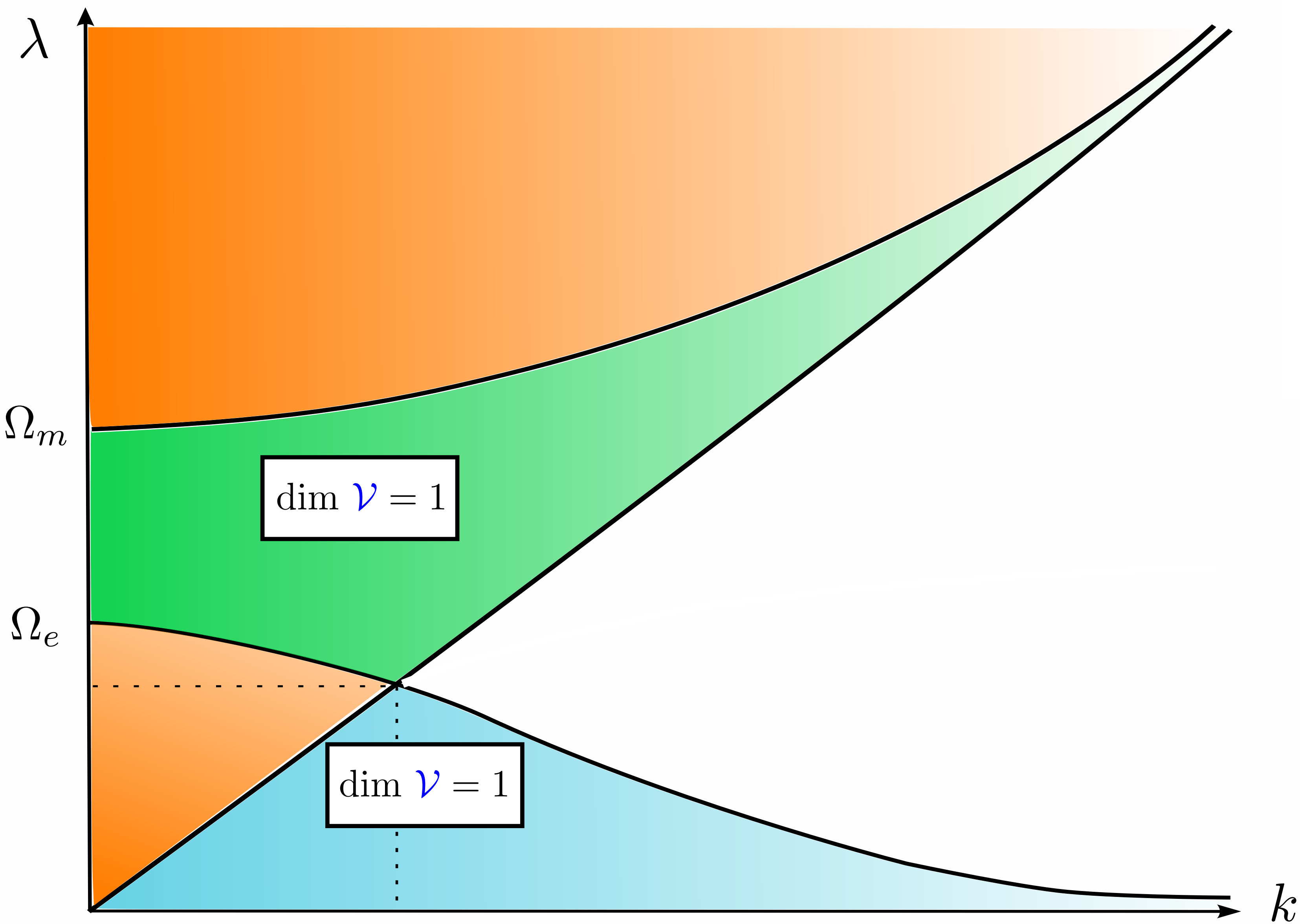

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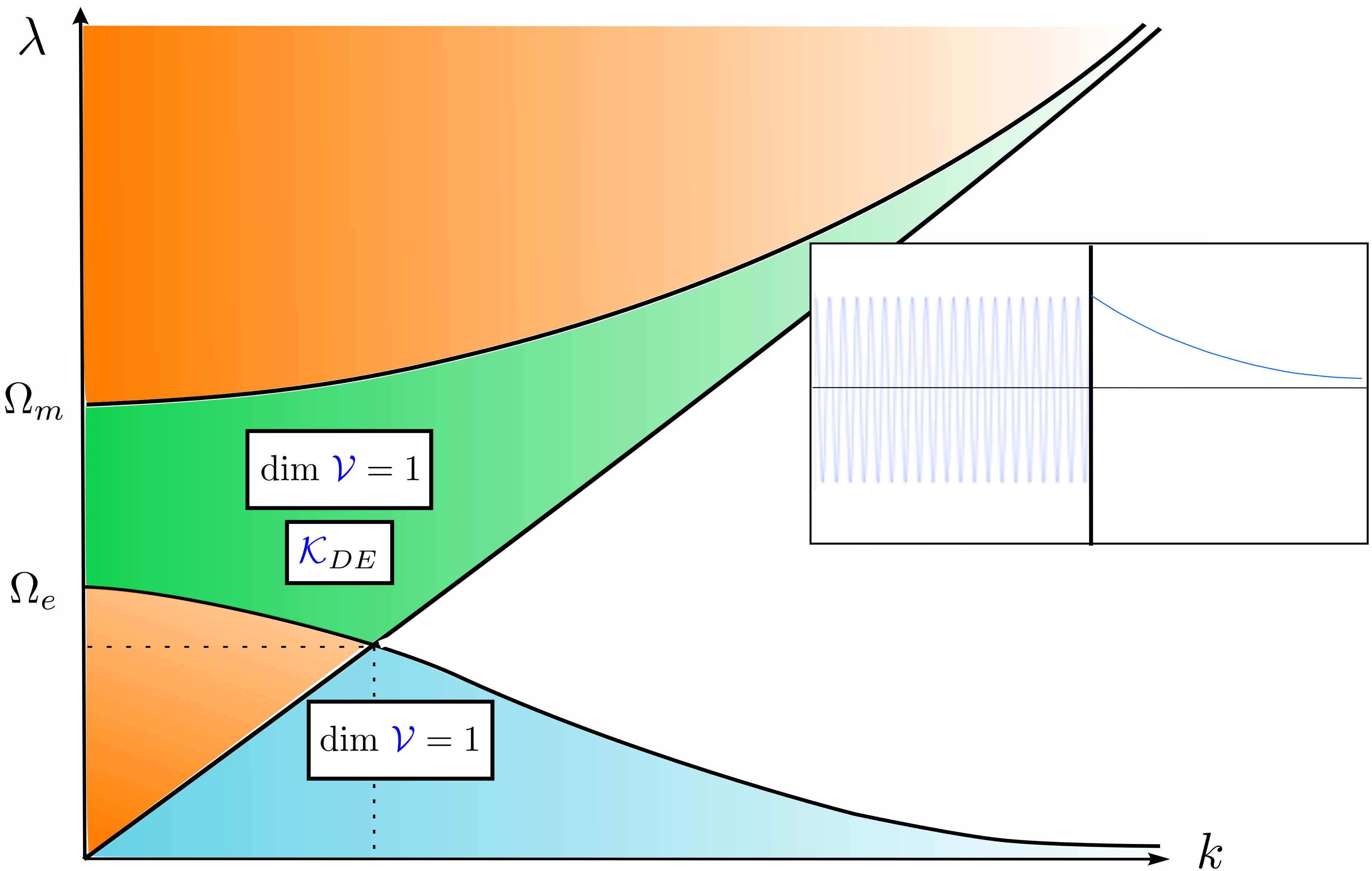

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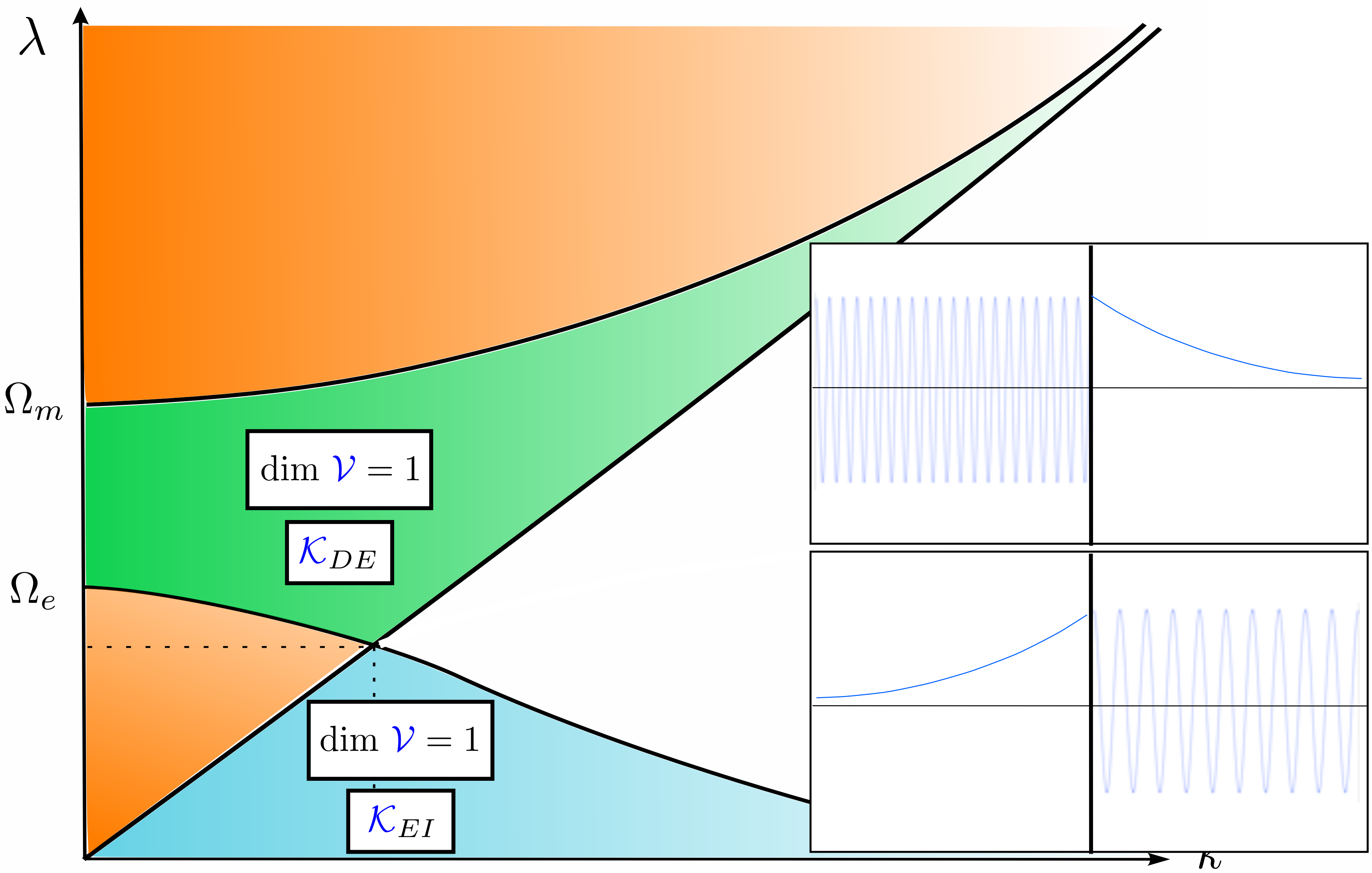

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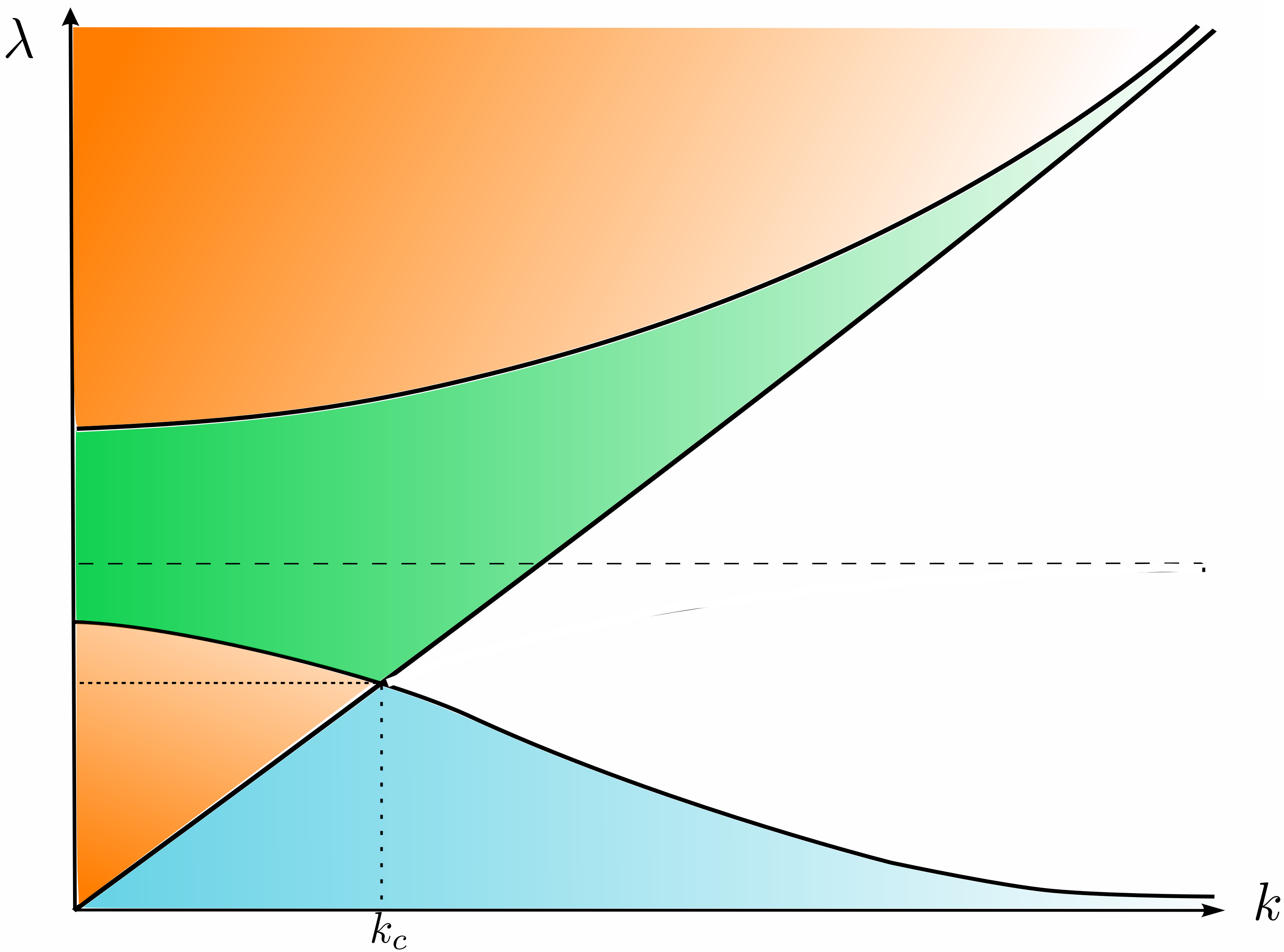

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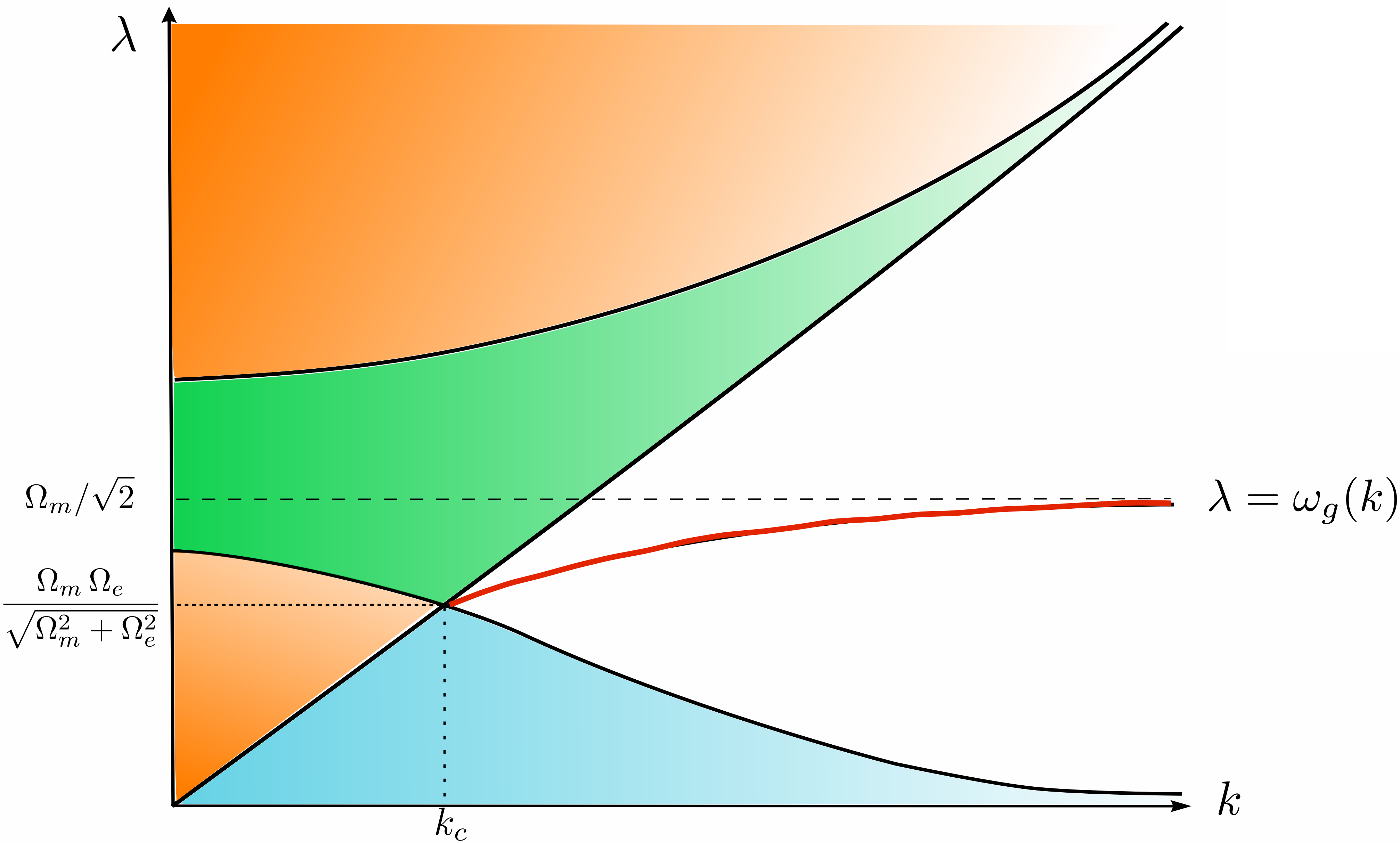

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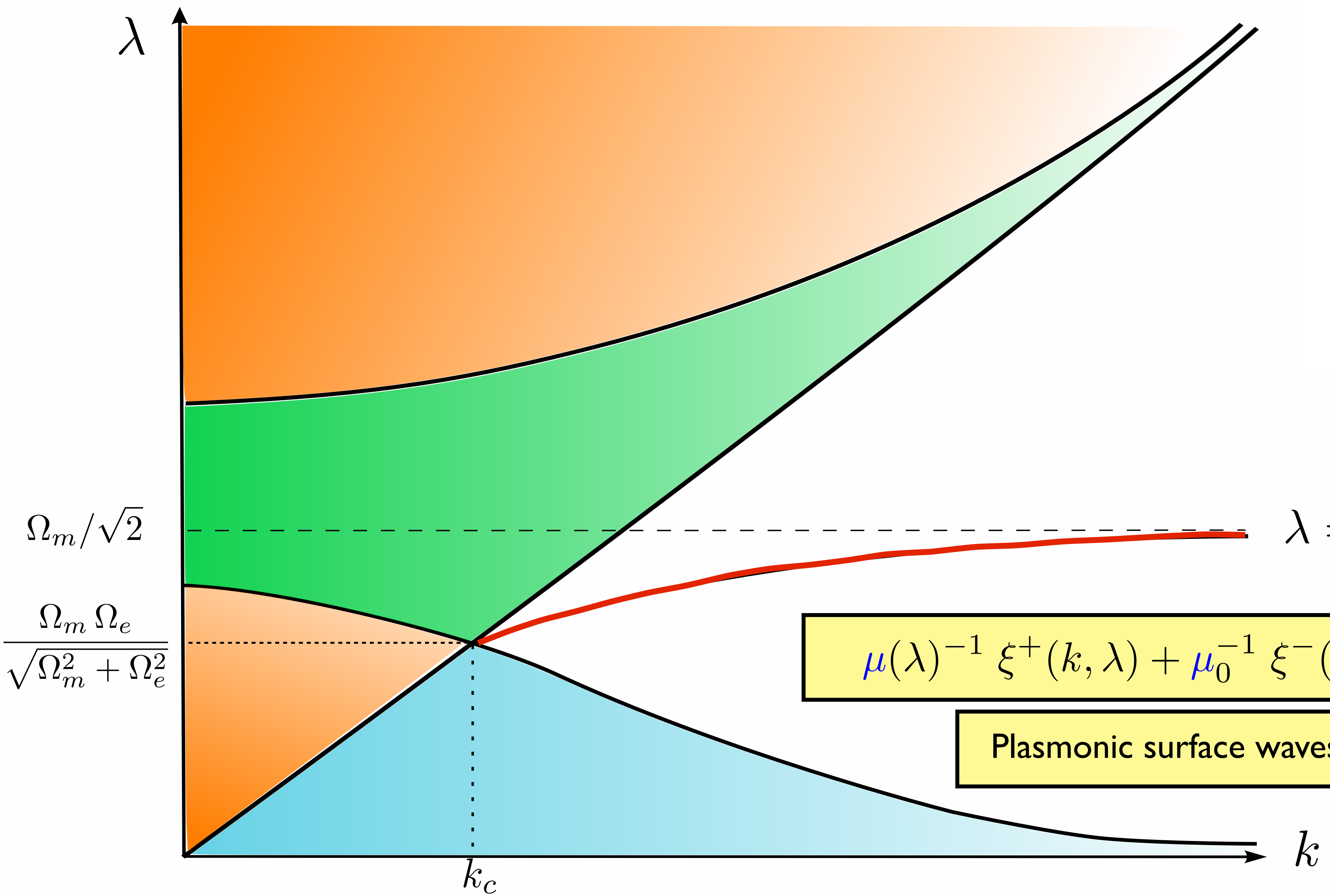

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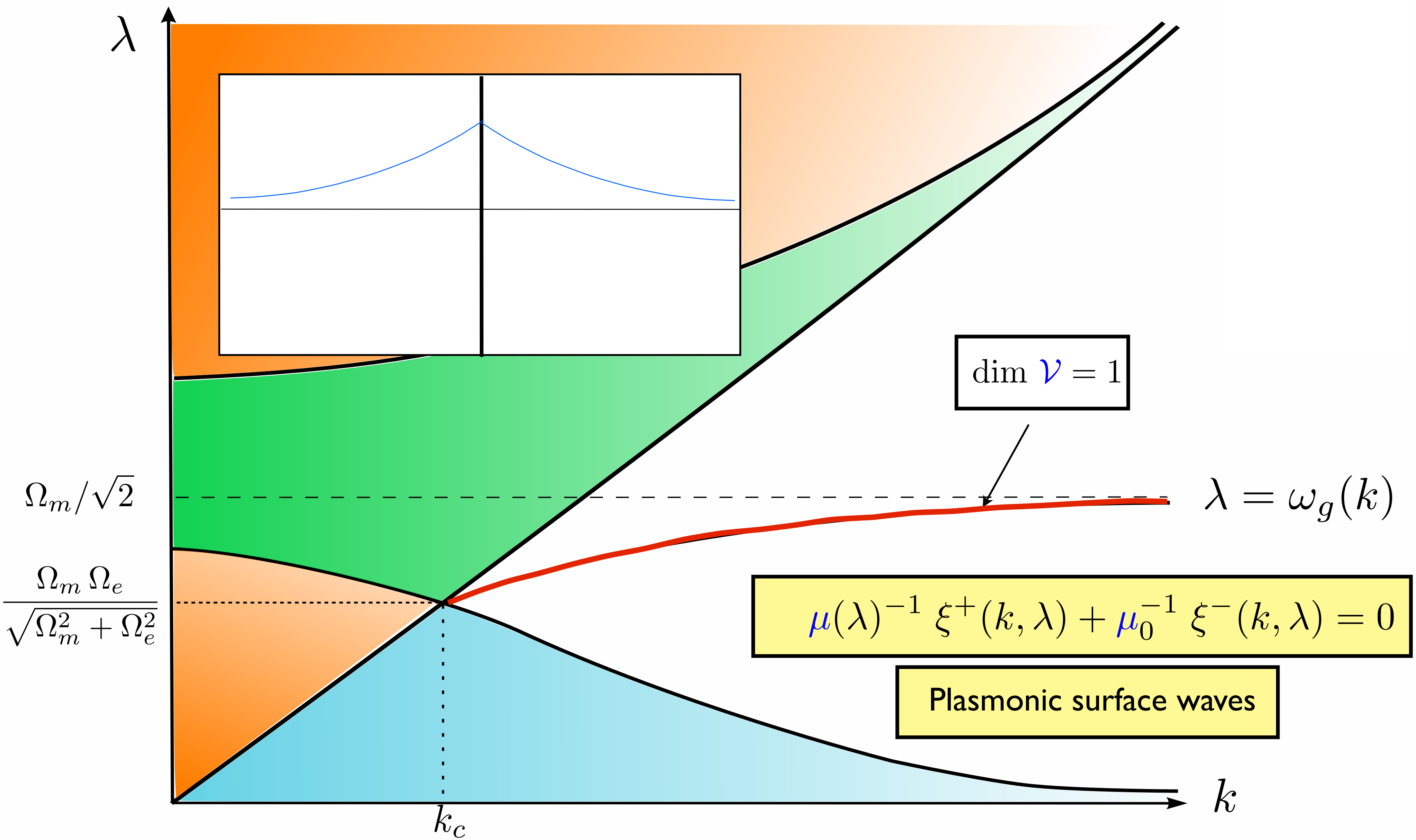

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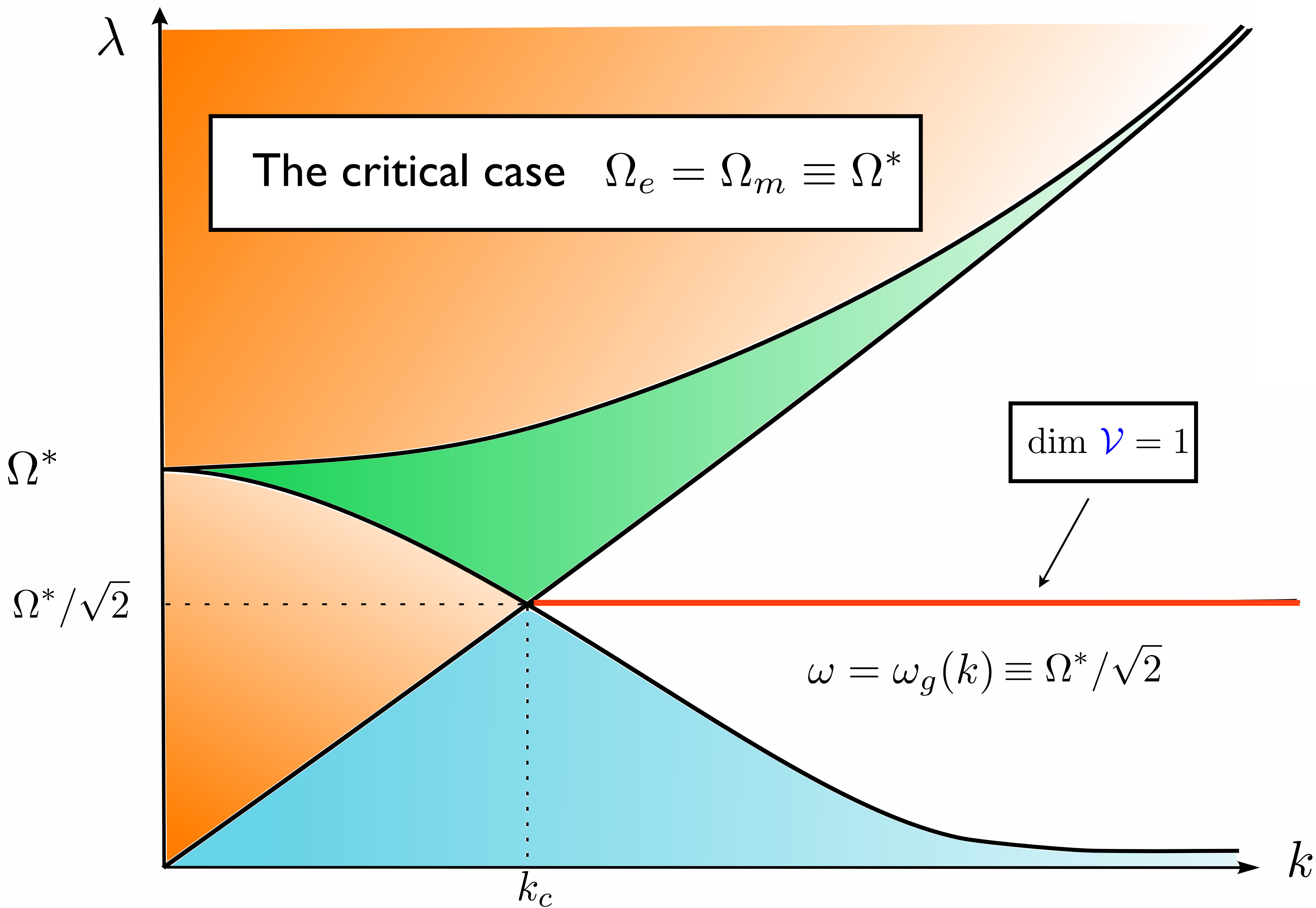

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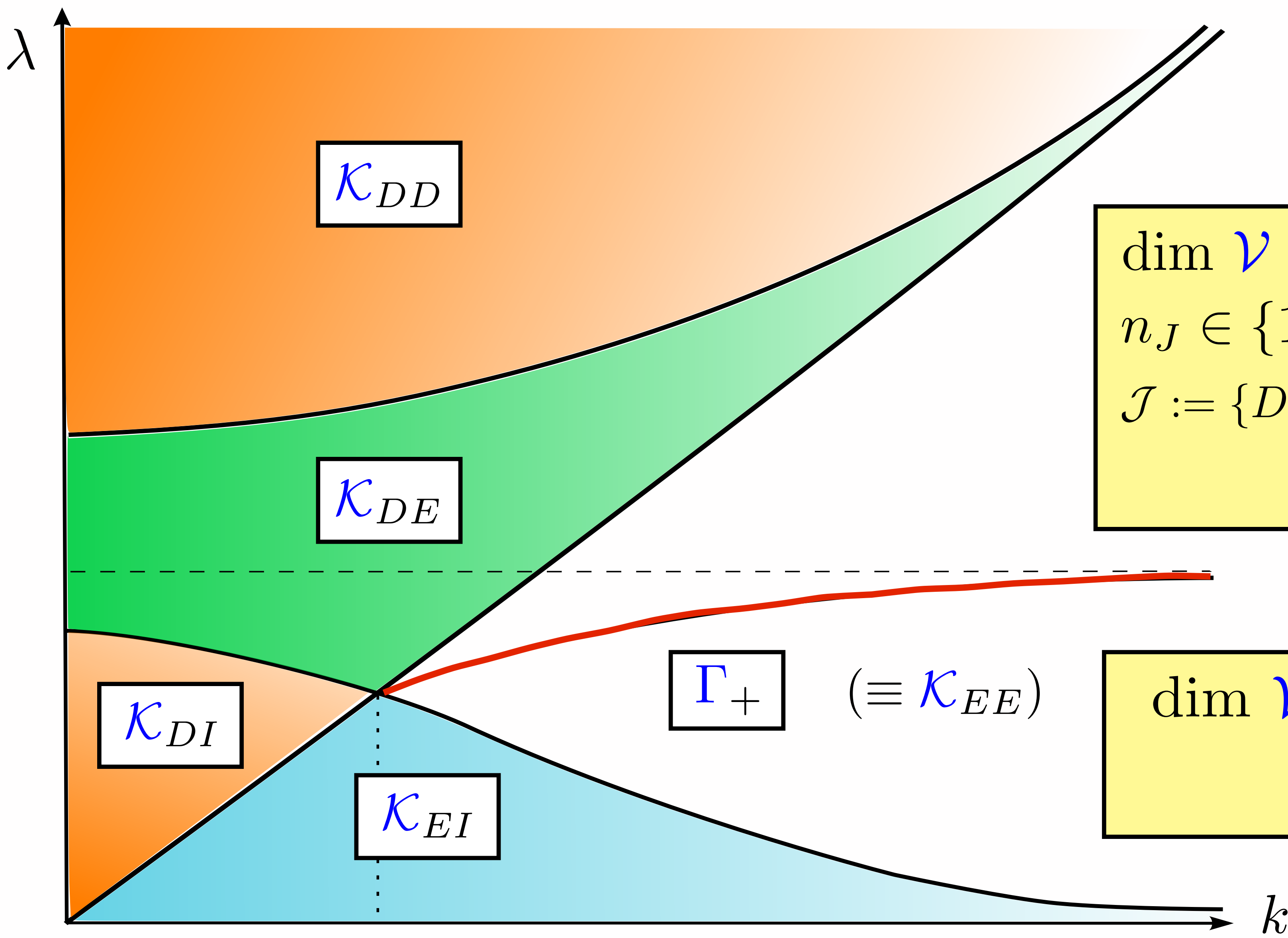

$\dim\,\mathcal{V} = n_J$ in $\mathcal{K}_J$

$n_J \in \{1,2\}$ $\quad J \in \mathcal{J}$

$\mathcal{J} := \{DD, DI, DE, EI\}$

$\dim\,\mathcal{V} = 1$ on Γ

Modal analysis

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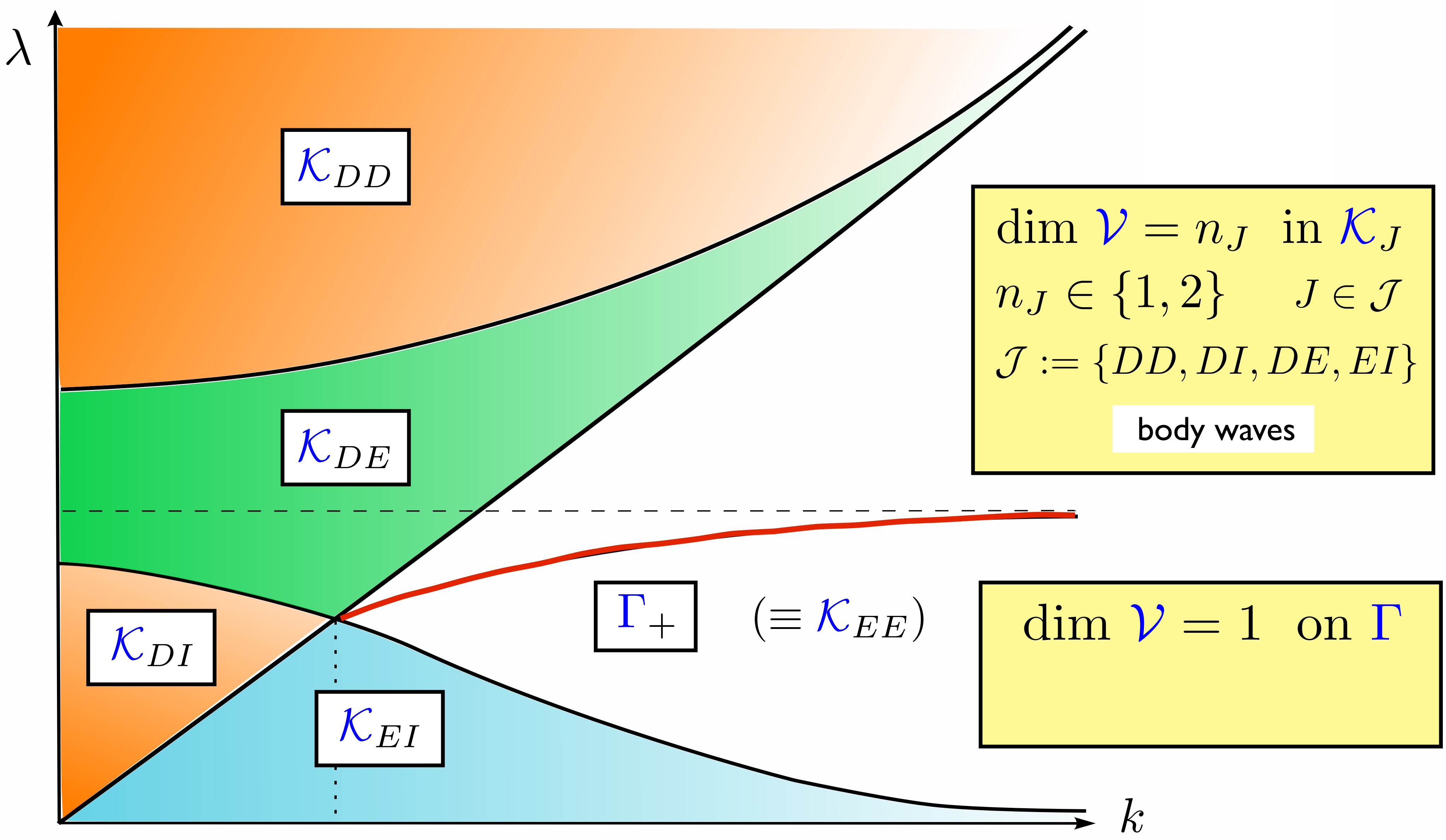

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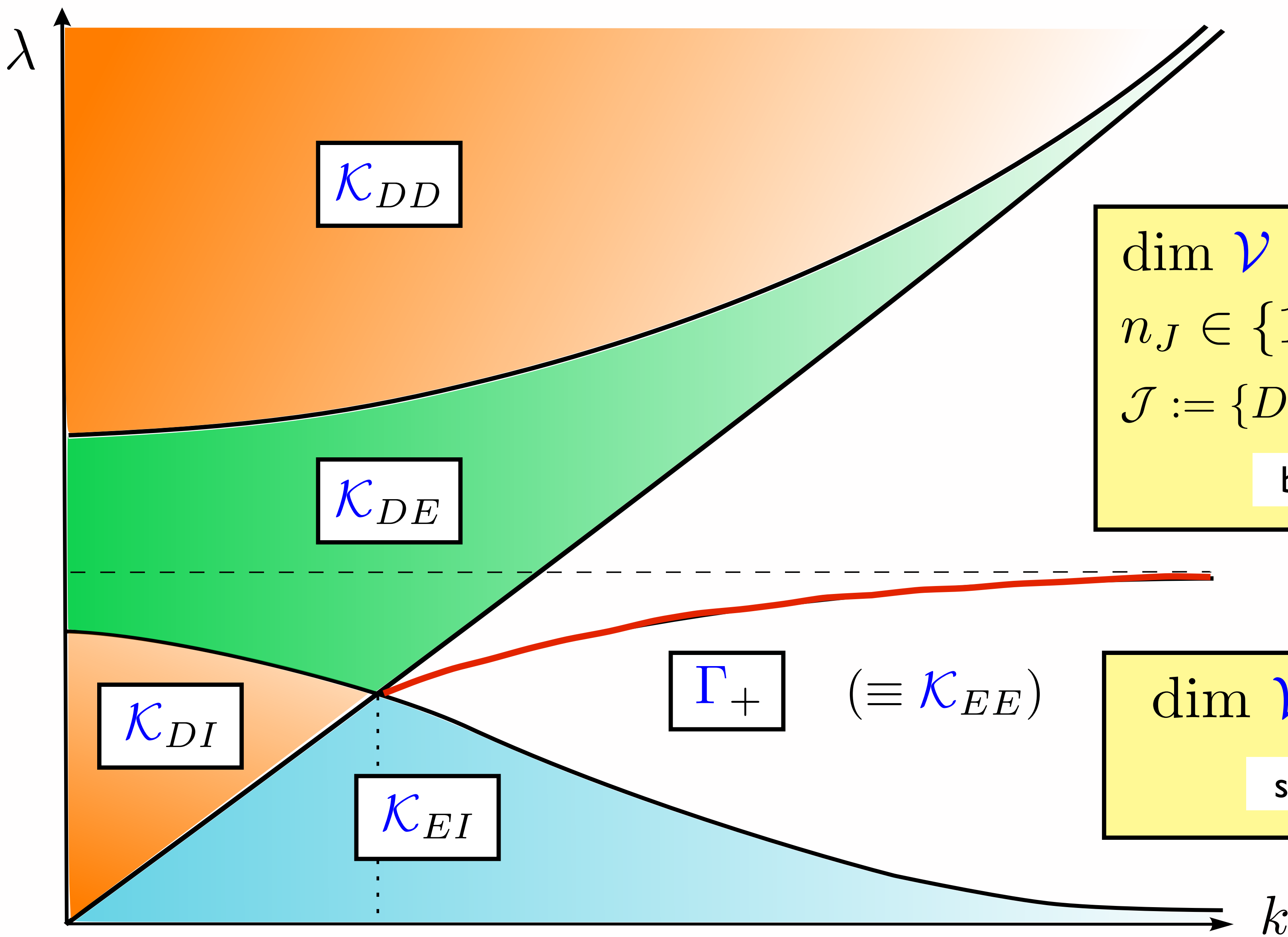


$\dim\,\mathcal{V} = n_J \quad \text{in } \mathcal{K}_J$

$n_J \in \{1,2\} \qquad J \in \mathcal{J}$

$\mathcal{J} := \{DD, DI, DE, EI\}$

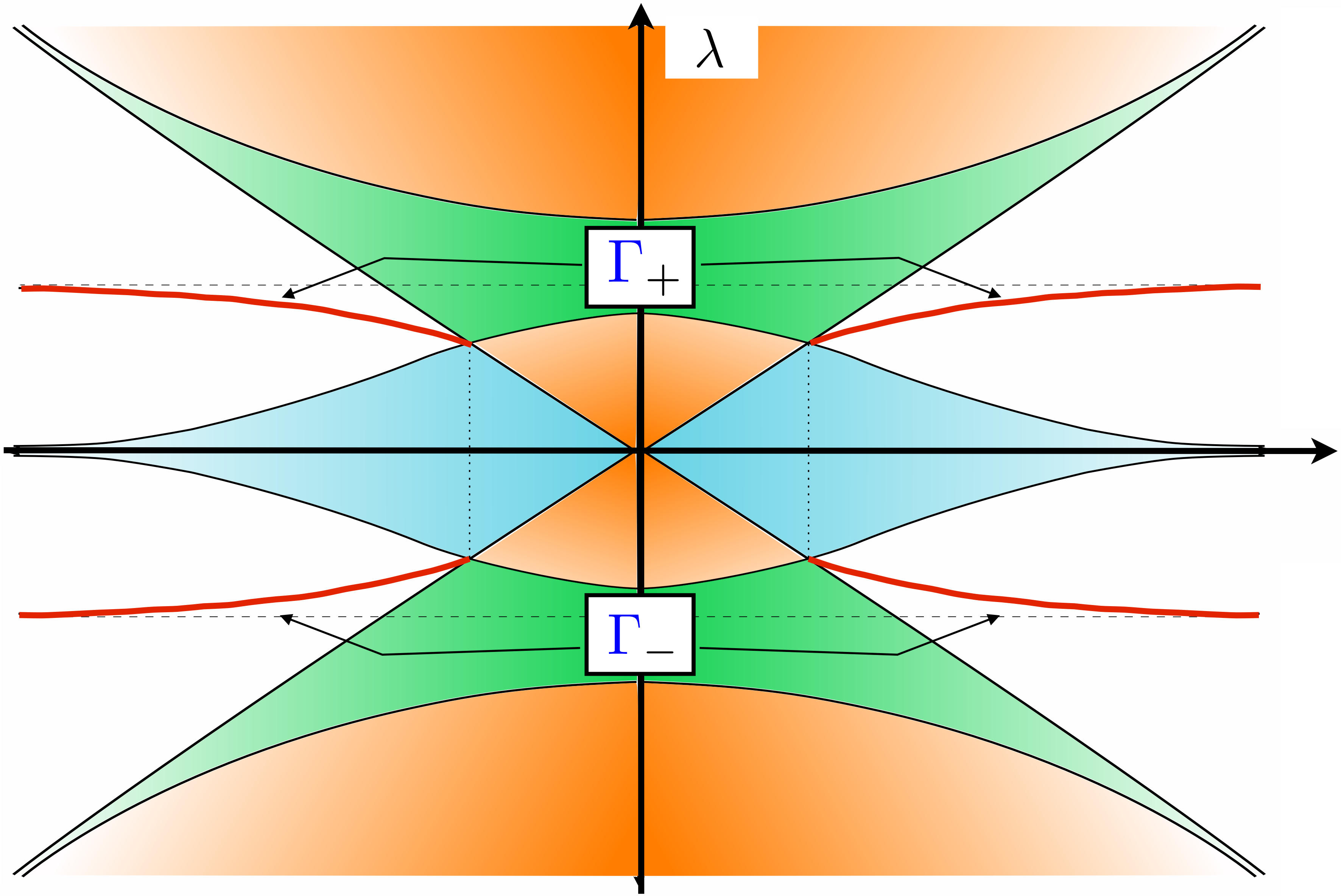
body waves

$\dim\,\mathcal{V} = 1 \quad \text{on } \Gamma$

surface waves

Modal analysis

$$\mathcal{V}(k,\lambda) := \{\mathbf{W} \,/\, \mathcal{A}\big(e^{iky}\,\mathbf{W}\big) = \lambda\,\big(e^{iky}\,\mathbf{W}\big)\,, |\mathbf{W}(x)| \leq C\,\big(1+|x|\big)\,\}$$



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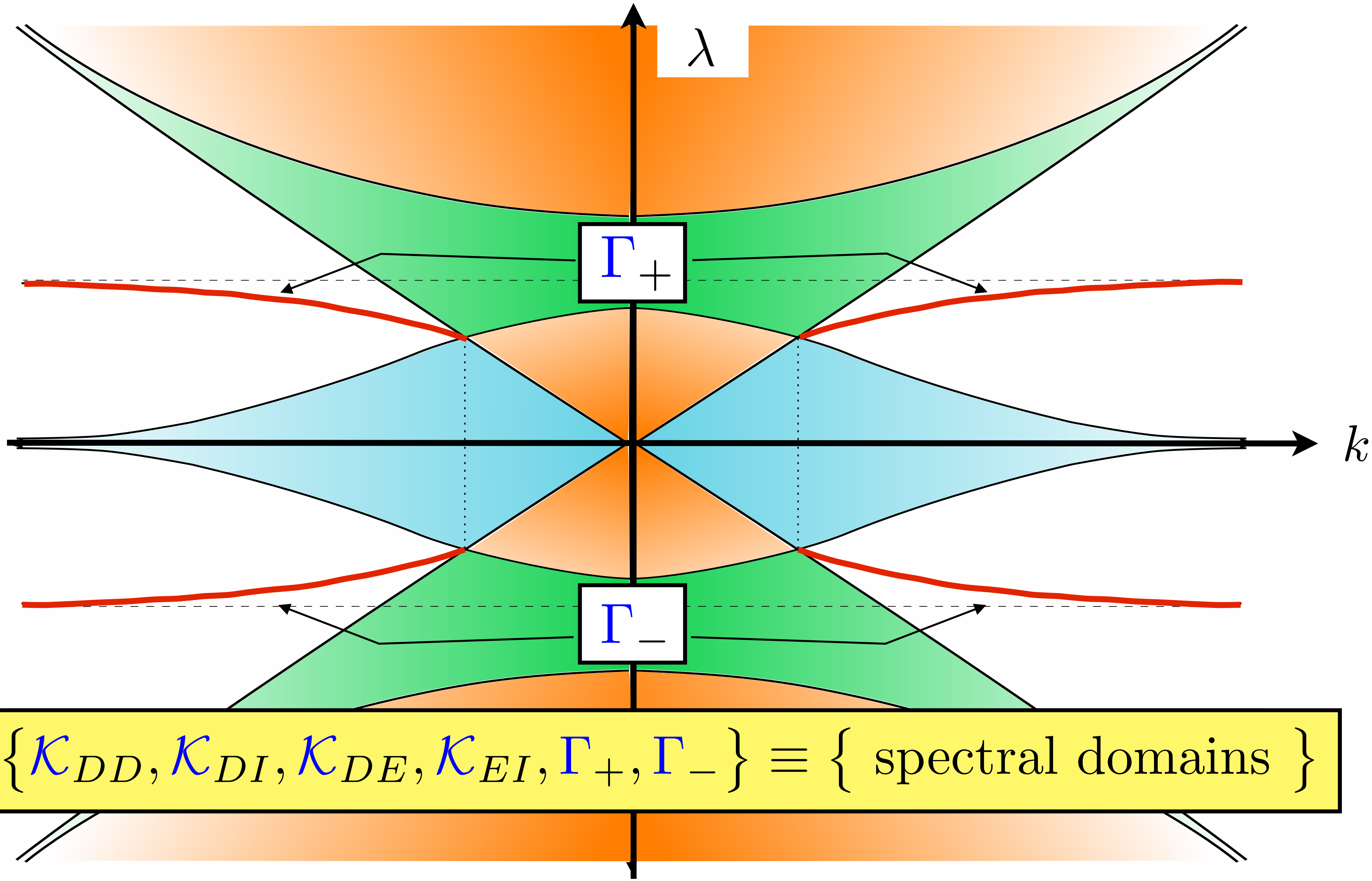

$$\{\mathcal{K}_{DD}, \mathcal{K}_{DI}, \mathcal{K}_{DE}, \mathcal{K}_{EI}, \Gamma_+, \Gamma_-\} \equiv \{\text{ spectral domains }\}$$

Modal analysis

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$$(k,\lambda) \text{ in } \mathcal{K}_J \quad \Longrightarrow \quad \mathcal{V}(\lambda,\omega) = \text{span}\,\big\{\mathbf{W}_{\ell,J}(k,\lambda;x), 1 \leq \ell \leq n_J\big\}$$

$$(k,\lambda) \text{ in } \Gamma_\pm \quad \Longrightarrow \quad \mathcal{V}(\lambda,\omega) = \text{span}\,\big\{\mathbf{W}_g^{\pm}(k\,;x)\big\}$$

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where the $\mathbf{W}_{\ell,J}(k,\omega;x)$ and $\mathbf{W}_g^\pm(k\,;x)$ are adequately chosen

$\displaystyle\int \big|\mathbf{W}_g^\pm(k\,;x)\big|_\rho^2\,dx = 1$ $\quad |\mathbf{U}|_\rho^2 := \varepsilon_0|E|^2 + \mu_0\,|\mathbf{H}|^2 + \Omega_e^2\,|\Phi|^2 + \Omega_m^2|\Psi|^2$ if $\mathbf{U} = (E,\mathbf{H},\Phi,\Psi)$

$\mathbf{W}_{\ell,J}(k,\lambda;x)$ is normalized according to Stone's theorem

Generalized Fourier transform

$$\mathcal{H} = L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2_+) \times L^2(\mathbb{R}^2_+)^2$$

Generalized Fourier transform

$$\mathcal{H} = L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2_+) \times L^2(\mathbb{R}^2_+)^2$$

$$\widehat{\mathcal{H}} = \bigoplus_{J \in \mathcal{J}} L^2(\mathcal{K}_J)^{n_J} \oplus_{\pm} \mathbf{L}^2(\Gamma_{\pm}, dk)$$

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$$\mathbf{U} := \big(E, \mathbf{H}, \mathbf{\Phi}, \mathbf{\Psi} \big) \longrightarrow \widehat{\mathbf{U}} := \big(\widehat{\mathbf{u}}_{\ell,J}, \widehat{\mathbf{u}}_g^{\pm}\big) \quad J \in \mathcal{J},\ 1 \leq \ell \leq n_J$$

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$$\widehat{\mathbf{u}}_{\ell,J}(k,\lambda) = \int \big(\mathbf{U}(x,y), \mathbf{W}_{\ell,J}(k,\lambda,x)\, e^{iky} \big)_* \, dx\, dy \qquad (k,\lambda) \text{ in } \mathcal{K}_J$$

$$\widehat{\mathbf{u}}_g^{\pm}(k) = \int \big(\mathbf{U}(x,y), \mathbf{W}_g^{\pm}(k;x)\, e^{iky} \big)_* \, dx\, dy \qquad |k| > k_c$$

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Generalized Plancherel's theorem and inversion formula

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Generalized Plancherel's theorem and inversion formula

$$\|\mathbf{U}\|_{\mathcal{H}}^2 = \sum_{J \in \mathcal{J}} \sum_{\ell=1}^{n_J} \int_{\mathcal{K}_J} |\widehat{\mathbf{u}}_{\ell,J}(\lambda, k)|^2 \, dk d\lambda + \sum_{\pm} \int_{|k|>k_c} |\widehat{\mathbf{u}}_g(k)|^2 \, dk$$

Generalized Fourier transform

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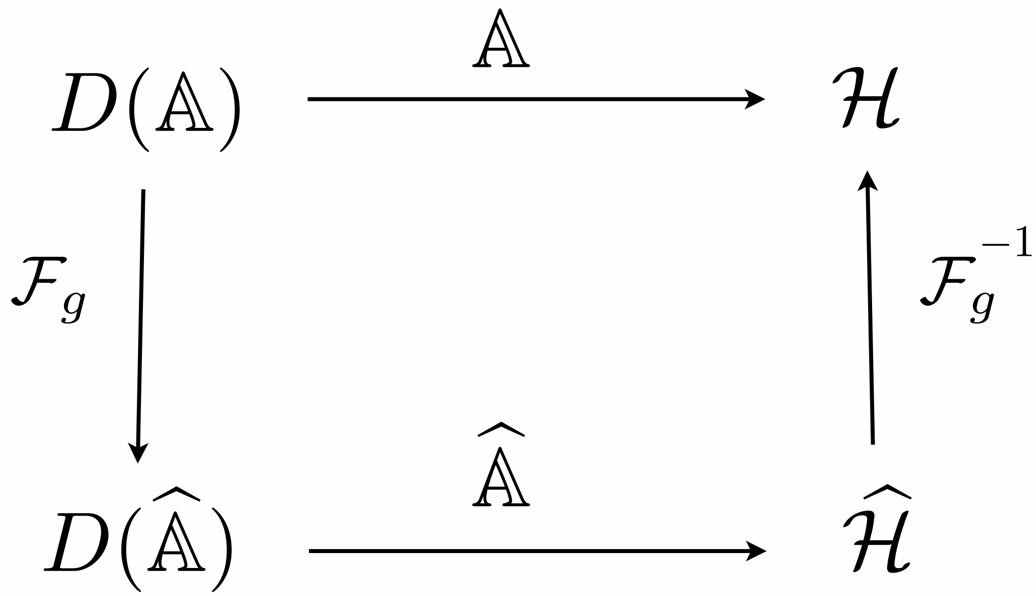
Diagonalization theorem



$$\widehat{\mathbf{U}} := \left(\widehat{\mathbf{u}}_{\ell,J}, \widehat{\mathbf{u}}_g^{\pm}\right) \longrightarrow \widehat{\mathbb{A}}\widehat{\mathbf{U}} \equiv \widehat{\mathbf{V}} = \left(\widehat{\mathbf{v}}_{\ell,J}, \widehat{\mathbf{v}}_g^{\pm}\right)$$

$$\widehat{\mathbf{v}}_{\ell,J}(k,\lambda) = \lambda\, \widehat{\mathbf{u}}_{\ell,J}(k,\lambda) \qquad \widehat{\mathbf{v}}_g^{\pm}(k) = \pm\, \omega_g(k)\, \widehat{\mathbf{u}}_g^{\pm}(k)$$

Diagonalization theorem

$$\begin{array}{ccc} D(\mathbb{A}) & \xrightarrow{\quad \mathbb{A} \quad} & \mathcal{H} \\ \Big\downarrow \mathcal{F}_g & & \Big\uparrow \mathcal{F}_g^{-1} \\ D(\widehat{\mathbb{A}}) & \xrightarrow{\quad \widehat{\mathbb{A}} \quad} & \widehat{\mathcal{H}} \end{array}$$

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Remark : $\widehat{\mathbf{v}}_g^{\pm}(k) = \lambda\, \widehat{\mathbf{u}}_g^{\pm}(k)$ along $\lambda = \pm\, \omega_g(k)$

Diagonalization theorem

$$\begin{array}{ccc} D(\mathbb{A}) & \xrightarrow{\quad \mathbb{A} \quad} & \mathcal{H} \\ \mathcal{F}_g \big\downarrow & & \big\downarrow \mathcal{F}_g \\ D(\widehat{\mathbb{A}}) & \xrightarrow{\quad \widehat{\mathbb{A}} \quad} & \widehat{\mathcal{H}} \end{array}$$

$$\widehat{\mathbf{U}} := \left(\widehat{\mathbf{u}}_{\ell,J}, \widehat{\mathbf{u}}_g^{\pm}\right) \longrightarrow \widehat{\mathbb{A}}\widehat{\mathbf{U}} \equiv \widehat{\mathbf{V}} = \left(\widehat{\mathbf{v}}_{\ell,J}, \widehat{\mathbf{v}}_g^{\pm}\right)$$

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If $\Omega_e \neq \Omega_m$, the spectrum of $\mathbb{A}$ is purely continuous.

If $\Omega_e = \Omega_m$, $\pm\,\omega_p$ is an eigenvalue of infinite multiplicity (eigenspace $\mathcal{H}_g^{\pm}$).

The representation theorem

$$\mathbf{U}(\cdot,t) = \sum_{J\in\mathcal{J}} \sum_{j=1}^{n_J} \mathbf{U}_{\ell,J}(\cdot,t) + \sum_{\pm} \mathbf{U}_g^{\pm}(\cdot,t)$$

$$\widehat{\mathbf{U}}(\cdot,t) = \big(\widehat{\mathbf{u}}_{\ell,J}(\cdot,t), \widehat{\mathbf{u}}_g(\cdot,t)\big)$$

$$\frac{\mathrm{d}\widehat{\mathbf{u}}_{\ell,J}}{\mathrm{d}t}(\lambda,k,t) - i\,\lambda\,\widehat{\mathbf{u}}_{\ell,J}(\lambda,k,t) = f_{\ell,J}(\lambda,k)\,e^{i\omega t}$$

$$\frac{\mathrm{d}\widehat{\mathbf{u}}_g^{\pm}}{\mathrm{d}t}(k,t) \mp \; i\,\omega_g(k)\;\widehat{\mathbf{u}}_g^{\pm}(k) = f_g^{\pm}(k)\,e^{i\omega t}$$

One gets a quasi-explicit representation of the solution

Long time analysis

$$\mathbf{U}(\cdot,t) = \sum_{J\in\mathcal{J}} \sum_{j=1}^{n_J} \mathbf{U}_{\ell,J}(\cdot,t) + \sum_{\pm} \mathbf{U}_g^{\pm}(\cdot,t)$$

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Although the study of the guided part $\mathbf{U}_g^{\pm}(\cdot,t)$ of the solution is easier than the rest, this is the part of the solution which leads to distinguish the critical and non critical cases.

Passage to the limit : non critical case

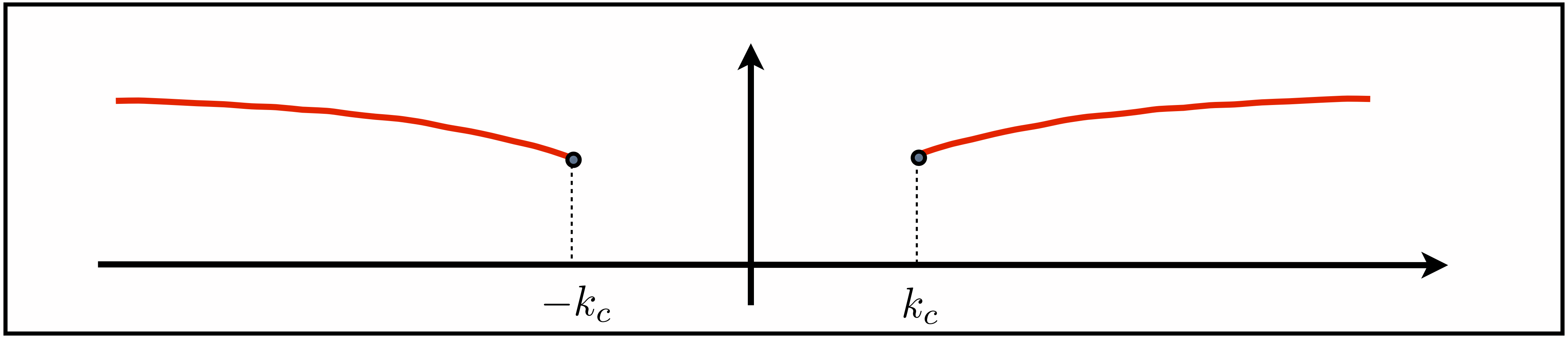
In this case, the explicit integration of

$$\frac{\mathrm{d}\widehat{\mathbf{u}}_g^+}{\mathrm{d}t}(k,t) - \ i\,\omega_g(k)\,\widehat{\mathbf{u}}_g^+(k) = f_g^+(k)\,e^{i\omega t}$$

leads to the following expression

$$\left(\mathbf{U}_g^+(\cdot,t),\mathbf{V}\right)_{\mathcal{H}} = e^{i\omega t}\int_{|k|>k_c} \frac{1-e^{i(\omega_g(k)-\omega)t}}{\omega_g(k)-\omega}\,\widehat{f}_g(k)\,\widehat{v}_g(k)\,dk$$

Passage to the limit : non critical case



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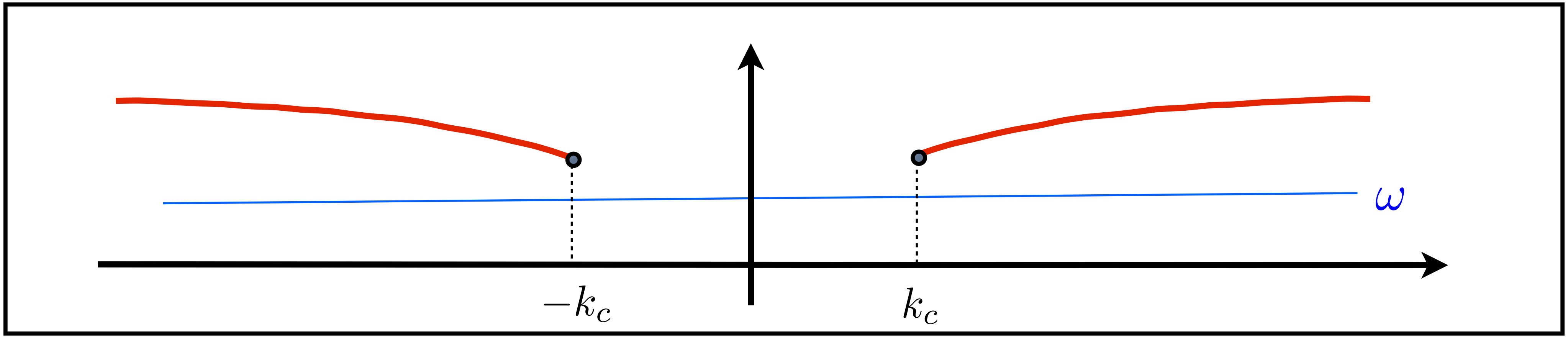

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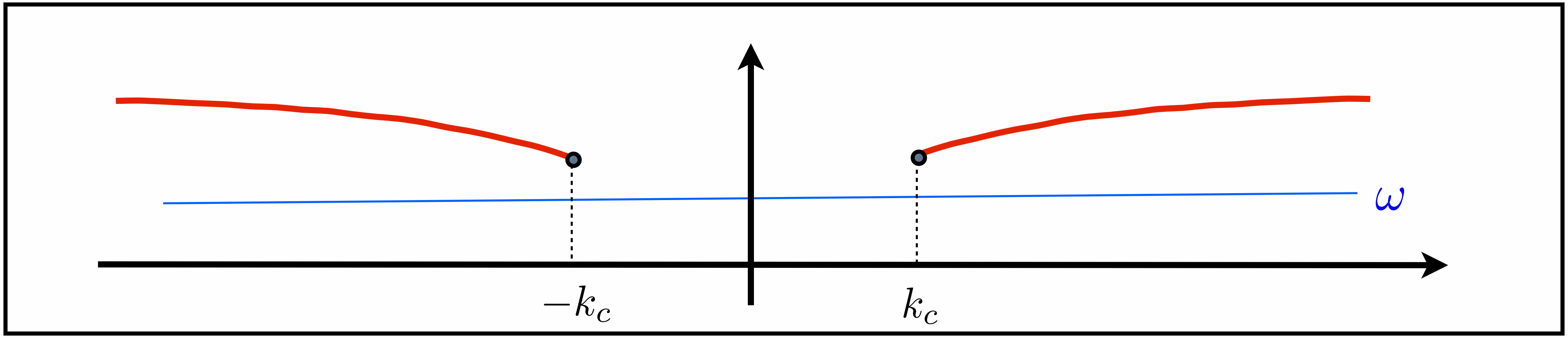

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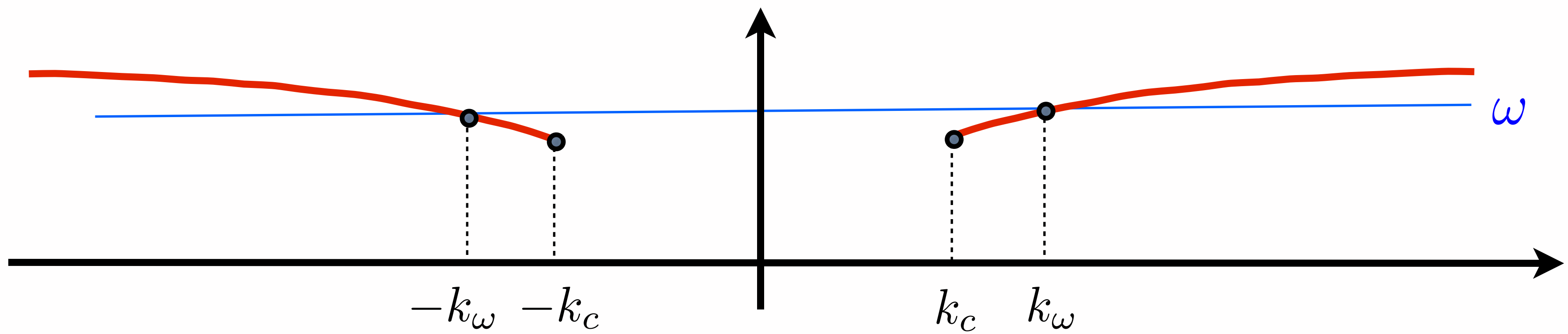
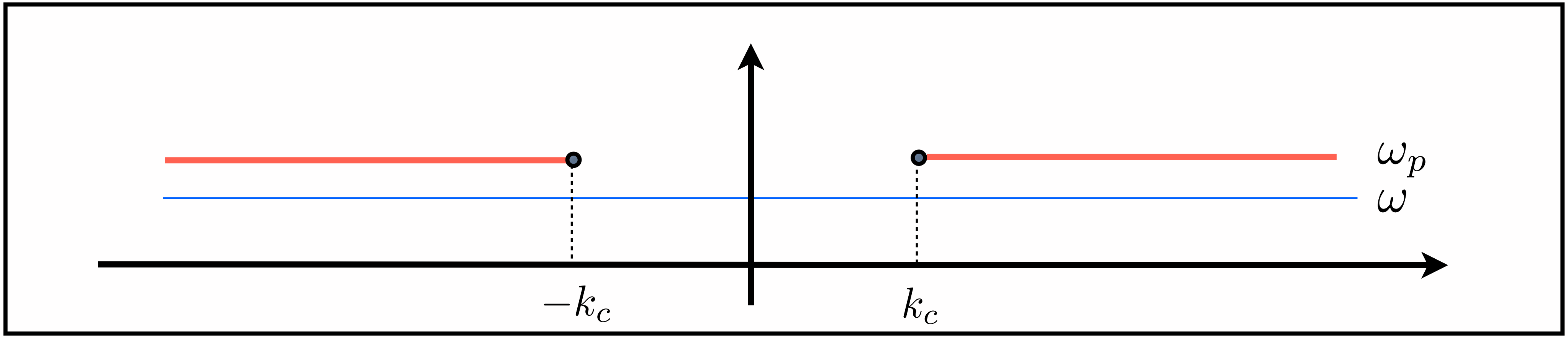


$$(\mathbf{U}_g^+(\cdot,t),\mathbf{V})_{\mathcal{H}} = e^{i\omega t}\int_{|k|>k_c} \frac{1-e^{i(\omega_g(k)-\omega)t}}{\omega_g(k)-\omega}\,\widehat{f}_g(k)\,\widehat{v}_g(k)\;dk$$

$$\lim_{t\to+\infty}\int_{|k|>k_c} \frac{1-e^{i(\omega_g(k)-\omega)t}}{\omega_g(k)-\omega}\,f_g(k)\,v_g(k)\;dk = P.V.\int_{|k|>k_c}\frac{\widehat{f}_g(k)\,\widehat{v}_g(k)}{\omega_g(k)-\omega}\,dk + i\pi\sum_{\pm}\frac{\widehat{f}_g(\pm k_\omega)\,\widehat{v}_g(\pm k_\omega)}{\omega_g'(\pm k_\omega)}$$

corresponds to the plasmonic surface waves observed in the first numerical simulation

Passage to the limit : the critical case

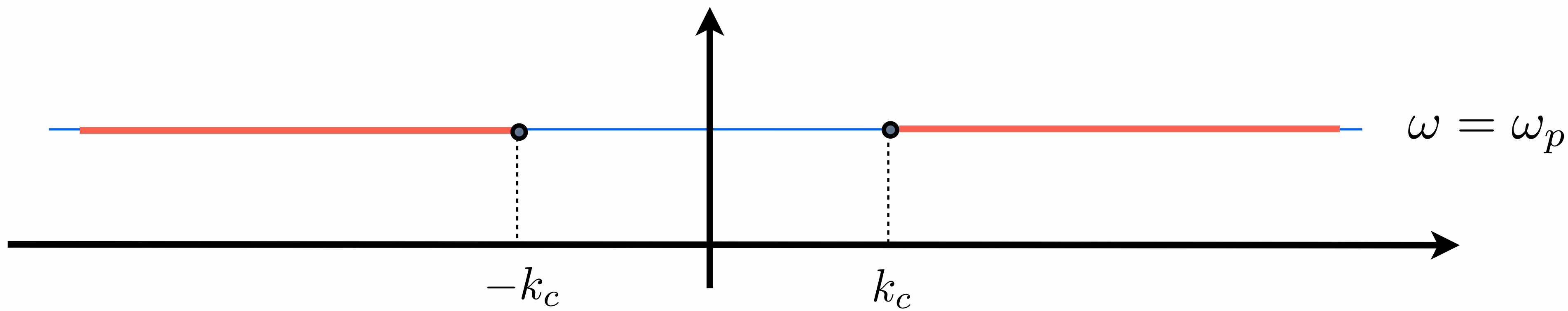


$$\frac{\mathrm{d}\widehat{\mathbf{u}}_g^+}{\mathrm{d}t}(k,t) - \; i\,\omega_p\,\widehat{\mathbf{u}}_g^+(k) = \widehat{f}_g^+(k)\,e^{i\omega t}$$

$$\left(\mathbf{U}_g^+(\cdot,t),\mathbf{V}\right)_{\mathcal{H}} = \frac{e^{i\omega t} - e^{i\omega_p t}}{\omega - \omega_p}\int_{|k|>k_c} \widehat{f}_g^+(k)\,\widehat{v}_g^+(k)\;dk$$

beat phenomenon

Passage to the limit : the critical case



$$\frac{\mathrm{d}\widehat{\mathbf{u}}_g^+}{\mathrm{d}t}(k,t) - \; i\,\omega_p\,\widehat{\mathbf{u}}_g^+(k) = \widehat{f}_g^+(k)\,e^{i\omega t}$$

$$\left(\mathbf{U}_g^+(\cdot,t), \mathbf{V}\right)_{\mathcal{H}} = i\;t\;e^{i\omega_p t}\int_{|k|>k_c} \widehat{f}_g^+(k)\,\widehat{v}_g^+(k)\;dk$$

linear blow-up

Conclusions and perspectives

This is a first contribution to the mathematical analysis of transmission problems with metamaterials.

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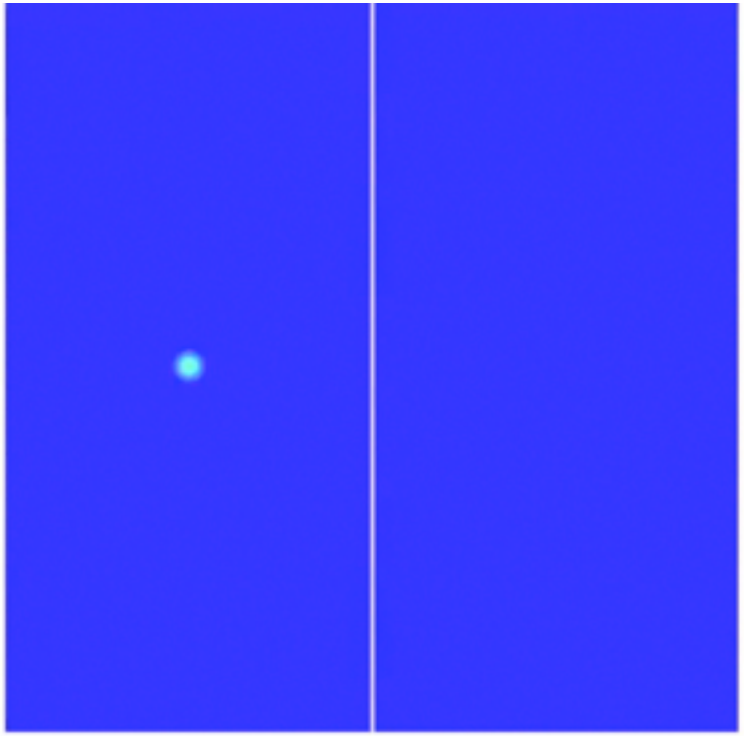

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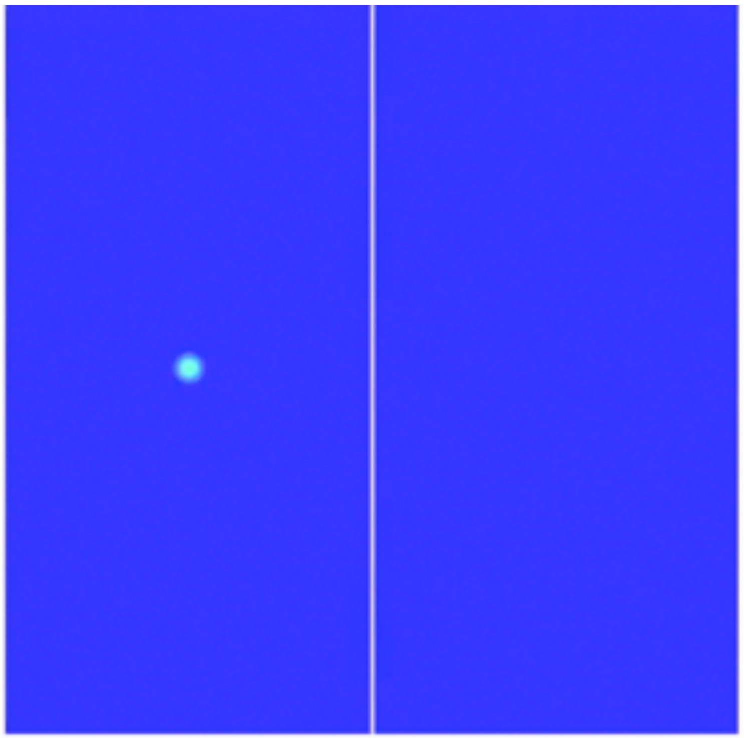

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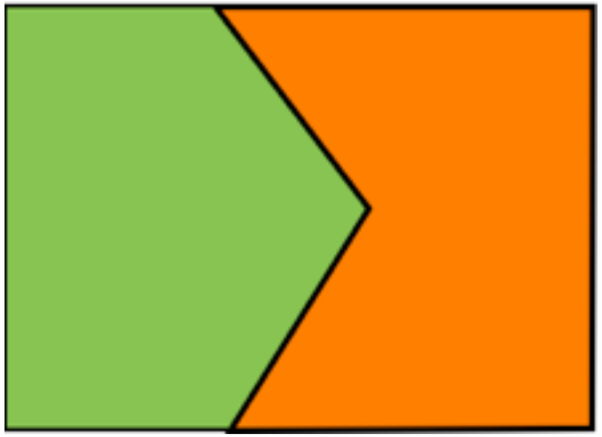
The present results can be generalized in 3D.
The case of a smooth curved interface is open.



More difficulties and new phenomena are expected with a non smooth interface (black hole phenomenon ?)