Plasma boundary reconstruction using topological asymptotic expansion

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joint work with

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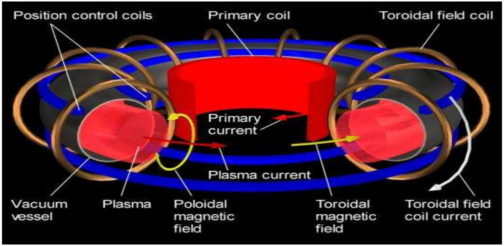
Plan

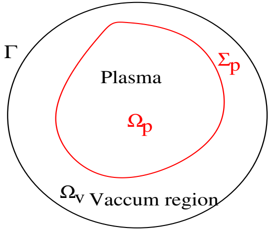
Introduction

The Tokamak: The Tokamak is an experimental machine which aims to confine the plasma in a magnetic field to control the nuclear fusion of atoms of mass law. The real-time reconstruction of the plasma magnetic equilibrium in a Tokamak is a key point to access high performance regimes.



The plasma equilibrium:

We denote by (r, φ, z) the three-dimentional cylindrical coordinates system. Since the tokamak is an axisymmetric troidal device, we may assume that all magnetic quantities do not depend on the troidal angle φ.



The plasma equilibrium may be studied in any cross section (r, z), named poloidal section.
It is described by the equation

$$L\psi = 0 \text{ in } \Omega_v$$

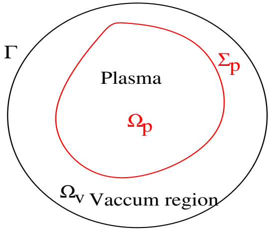
$-$ Ω_v is the domain included between the tokamak boundary Γ and the plasma boundary Σ, called the vacuum region.
$-$ ψ is the poloidal magnetic flux.
$-$ $\mathcal{L}$ is the Grad-Shafranov operator

$$\mathcal{L} = -\frac{\partial}{\partial r}\left(\frac{1}{\mu r}\frac{\partial}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{\mu r}\frac{\partial}{\partial z}\right)$$

The Tokamak problem : We consider here the inverse problem of determining plasma boundary Σ_p location from over-specified boundary data on Γ.

Knowing a complete set of Cauchy data, the poloidal flux ψ satisfies the system

$$\begin{cases} L\psi = 0 & \text{in} \quad \Omega\backslash\overline{\Omega_p}, \\ \dfrac{1}{r}\dfrac{\partial\psi}{\partial n} = \Phi & \text{on} \quad \Gamma, \\ \psi = \psi_m & \text{on} \quad \Gamma, \\ \psi = 0 & \text{on} \quad \Sigma_p. \end{cases}$$



− Ω is the domain limited by the boundary Γ,
− Φ is the magnetic field and ψ_m is the measured poloidal flux on Γ.

In this formulation the domain Ω is unknown since the free plasma boundary Σ_p is unknown. This problem is ill posed in the sense of Hadamard.

Formulation of the problem: In order to determine the unknown plasma boundary Σ_p location we propose two formulations for the considered inverse problem.

The first formulation : consists in finding the optimal location of the plasma boundary Σ_P minimizing the cost function

$$T(\psi, \Sigma_p) := \int_\Gamma |\psi - \psi_m|^2 \, \mathrm{ds}$$

where ψ is the solution to

$$\begin{cases} L\psi &= 0 & \text{in } \Omega \backslash \overline{\Omega_p}, \\ \dfrac{1}{r}\dfrac{\partial \psi}{\partial n} &= \Phi & \text{on } \Gamma, \\ \psi &= 0 & \text{on } \Sigma_p. \end{cases}$$

The Kohn-Vogelius cost function : For any plasma domain Ω_p, we define two forward problems:
the first one is associated to the magnetic field Φ (Newmann datum):

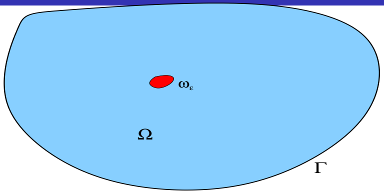
$$(\mathcal{P}_N)\begin{cases} L\psi_N = 0 & \text{in } \Omega\backslash\overline{\Omega_p} \\ \frac{1}{r}\frac{\partial\psi_N}{\partial n} = \Phi & \text{on } \Gamma \\ \psi_N = 0 & \text{on } \Sigma_p. \end{cases}$$

the second one is associated to the measured poloidal flux ψ_m(Dirichlet datum)

$$(\mathcal{P}_D)\begin{cases} L\psi_D = 0 & \text{in } \Omega\backslash\overline{\Omega_p} \\ \psi_D = \psi_m & \text{on } \Gamma \\ \psi_D = 0 & \text{on } \Sigma_p. \end{cases}$$

The identification process is based on the minimization of the following energy function

$$\mathcal{K}(\Omega\backslash\overline{\Omega_p}) = \int_{\Omega\backslash\overline{\Omega_p}} \frac{1}{r}|\nabla\psi_D - \nabla\psi_N|^2 \,\mathrm{dx}.$$



Topological gradient method:

Main idea : studying the variation of the design function $\mathcal{J}$ with respect to the creation of a small hole ω_ε in Ω.

It leads to an asymptotic expansion of the form

$$\mathcal{J}(\Omega\backslash\overline{\omega_\varepsilon}) - \mathcal{J}(\Omega) = f(\varepsilon)\delta\mathcal{J}(z) + o(f(\varepsilon)).$$

where

- $f(\varepsilon)$: is a scalar function known explicitly and goes to zero with ε $\lim_{\varepsilon\to 0} f(\varepsilon) = 0$.
- $\delta\mathcal{J}$: topological gradient, called also topological sensitivity.

In order to minimize the cost function, the best location to insert a small hole in Ω is where $\delta\mathcal{J}$ is most negative.

In fact if $\delta\mathcal{J}(z) < 0$, we have $\mathcal{J}(\Omega\backslash\overline{\omega_\varepsilon}) < \mathcal{J}(\Omega)$ for small ε.

History :

✘ It has been introduced by Schumacher [1995] as “numerical approach” in structural mechanics using circular holes and Neumann b.c.

✘ Sokolowski [1999]: extended this idea to more general function using the adjoint method (case circular holes and Neumann b.c.).

✘ Masmoudi [2001]: introduced the Dirichlet condition case and given a more general approach to compute the topological gradient.

More recently, it has been generalized for different PDE: Elasticity, Laplace, Stokes, Helmholtz, Maxwell, Navier-Stokes, ...

Example: The Laplace operator admits an asymptotic expansion on the form

$$\mathcal{J}(\Omega \backslash \overline{\omega_\varepsilon}) - \mathcal{J}(\Omega) = f(\varepsilon)\delta\mathcal{J}(z) + o(f(\varepsilon)).$$

The topological gradient $\delta\mathcal{J}$ and the scalar function $f(\varepsilon)$ are described by the following table

B.C. on $\partial\omega_\varepsilon$	Topo. gradient $\delta\mathcal{J}$	function $f(\varepsilon)$
2D Dirichlet	$4\pi u_0(z)\, v_0(z) + \delta J$	$-1/\log(\varepsilon)$
3D Dirichlet	$6\pi u_0(z)\, v_0(z) + \delta J$	ε
2D Neumann	$-2\pi\nabla u_0(z).\nabla v_0(z) + \delta J$	ε^2
3D Neumann	$-2\pi\nabla u_0(z).\nabla v_0(z) + \delta J$	ε^3

- u_0 : solution to the Laplace operator, computed in the non perturbed domain.
- v_0 : solution to the adjoint problem, computed in the non perturbed domain.
- the term δJ depends on the considered cost function.

It has been successfully used for various applications:

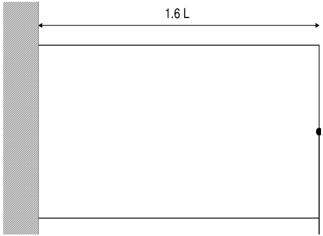
Structural mechanics:

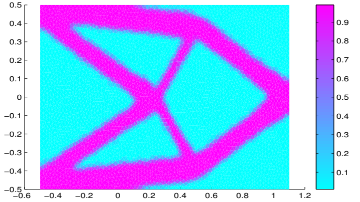
Guarreau, Guillaume, Masmoudi (2001)

Maximization of the compliance for a 2D cantilever beam: The initial domain is a plain rectangle with one edge clamped and a pointwise load is applied to the middle of the opposite edge.

Aim : find the optimal domain with a volume less than 40% of the initial one.

Idea : sensitivity analysis w.r.t. remove small part of the domain





Topological Sensitivity analysis

Image restoration: based on the edges detection.
Idea : edges are considered as a set of small craks.

Jaafar, Jaoua, Masmoudi, Siala (2006)

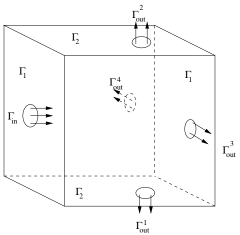
Lena example: Original image, Noisy image(20% g. noise) Restorated image

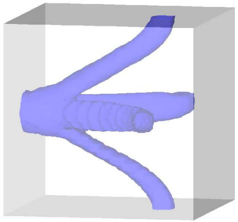
Fluid mechanics:

Hassine, Abdelwahed, Masmoudi (2008)

Aim : Find the optimal shape design of the tubes that connect the inlet to the outlets of the cavity minimizing the dissipated power in the fluid.

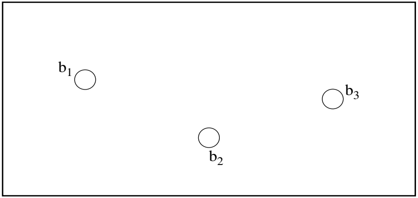
Idea : sensitivity analysis w.r.t. inserting a small obstacle in the fluid flow



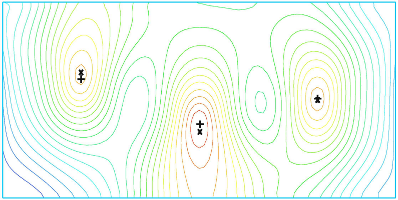


Main features of the topological gradient method:

1. No initial guess, with a priori poor information on the optimal shape.
2. It depends on solutions computed on the safe body.
3. It is fast and easy to be implemented.
4. The flaws location are characterized as the local minima of the topological gradient δj.



Exact locations



Isovalues of δj

Sensitivity analysis for the tracking function: Let ω_ε be a small geometry perturbation inside the domain Ω with a Dirichlet boundary condition on $\partial\omega_\varepsilon$. The tracking function is defined by

$$T(\Omega \backslash \overline{\omega_\varepsilon}) = \int_\Gamma |\psi_\varepsilon - \psi_m|^2$$

where ψ_ε satisfies the problem

$$\begin{cases} L\psi_\varepsilon & = & 0 & \text{in} & \Omega \backslash \overline{\omega_\varepsilon}, \\ \dfrac{1}{r}\dfrac{\partial \psi \varepsilon}{\partial n} & = & \Phi & \text{on} & \Gamma, \\ \psi_\varepsilon & = & 0 & \text{on} & \partial\omega_\varepsilon. \end{cases}$$

Theorem: If $\omega_\varepsilon = X_0 + \varepsilon\omega \subset \Omega$ where $X_0 \in \Omega$, $\varepsilon > 0$ and $\omega \subset \mathbb{R}^2$ is a given, regular and bounded domain containing the origin,
the tracking function T admits the following asymptotic expansion

$$T(\Omega \backslash \overline{\omega_\varepsilon}) = T(\Omega) - \frac{1}{\log(\varepsilon)} \frac{2\pi}{x_0} \psi_0(X_0)\ \phi_0(X_0) + o(-\frac{1}{\log(\varepsilon)})$$

where ϕ_0 is the solution to the associated adjoint problem.

The topological sensitivity analysis for the Kohn-Vogelius function:
the Kohn-Vogelius function $\mathcal{K}$ is defined by

$$\mathcal{K}(\Omega \setminus \overline{\omega_\varepsilon}) = \int_{\Omega\setminus\overline{\omega_\varepsilon}} \frac{1}{r} \left|\nabla\psi_N^\varepsilon - \nabla\psi_D^\varepsilon\right|^2 dx,$$

with ψ_N^ε and ψ_D^ε are the solutions to the Neumann and Dirichlet perturbed problems

$$(\mathcal{P}_N^\varepsilon) \begin{cases} L\psi_N^\varepsilon = 0 & \text{in } \Omega\setminus\overline{\omega_\varepsilon} \\ \frac{1}{r}\nabla\psi_N^\varepsilon \mathbf{n} = \Phi & \text{on } \Gamma \\ \psi_N^\varepsilon = 0 & \text{on } \partial\omega_\varepsilon, \end{cases} \qquad (\mathcal{P}_D^\varepsilon) \begin{cases} L\psi_D^\varepsilon = 0 & \text{in } \Omega\setminus\overline{\omega_\varepsilon} \\ \psi_D^\varepsilon = \psi_m & \text{on } \Gamma \\ \psi_D^\varepsilon = 0 & \text{on } \partial\omega_\varepsilon. \end{cases}$$

Theorem: The function $\mathcal{K}$ admits the following asymptotic expansion

$$\mathcal{K}(\Omega \setminus \overline{\omega_\varepsilon}) = \mathcal{K}(\Omega) + \frac{-2\pi}{log(\varepsilon)}\frac{1}{x_0}\left[\left|\psi_N^0(X_0)\right|^2 - \left|\psi_D^0(X_0)\right|^2\right] + o(\frac{-1}{log(\varepsilon)}). \qquad (1)$$

For a given small perturbation $\omega_z = z + \varepsilon\omega \subset \Omega$, we will study the variation of the function $\Delta_z(\varepsilon)$

$$\Delta_z(\varepsilon) = j(\Omega_{z,\varepsilon}) - j(\Omega) - \delta j(z), \ z \in \Omega$$

with respect to ε.
We expect to prove numerically that the function Δ_z satisfies the estimate

$$\Delta_z(\varepsilon) = o(\frac{-1}{\log(\varepsilon)}).$$

Denoting by β the parameter describing the behaviour of $\Delta_z(\varepsilon)$ with respect to $\frac{-1}{\log(\varepsilon)}$, i.e.

$$|\Delta_z(\varepsilon)| = O(|\frac{-1}{\log(\varepsilon)}|^{\beta}).$$

Then, β can be characterized as the slope of the line approximating the variation of the function $\varepsilon \mapsto \log(|\Delta_z(\varepsilon)|)$ with respect to $\log(-\log(\varepsilon))$.

Numerical validation of the theoretical asymptotic expansion

The tracking function : three locations $\omega^i = X_0^i + \varepsilon B(0,1) \subset \Omega$, $i = 1, 2, 3$ have been tested, where $\Omega = B(X_0, 1)$ with $X_0 = (2, 0)$.

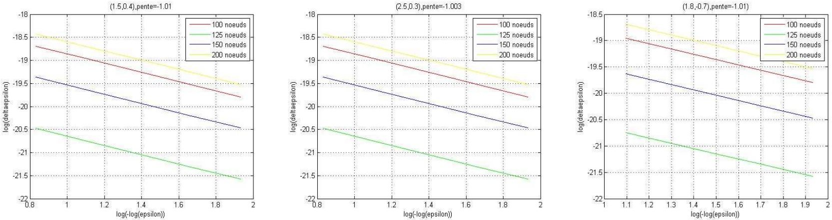


Figure: Variation of $\log(|\Delta_{z_i}(\varepsilon)|)$, $i = 1, ..., 3$ with respect to $\log(-\log(\varepsilon))$.

The perturbation	ω^1	ω^2	ω^3
Location X_0^i	$X_0^1 = (1.5, 0.4)$	$X_0^2 = (2.5, 0.3)$	$X_0^3 = (1.8, -0.7)$
The slope β_i	$\beta_1 = -1.01$	$\beta_2 = -1.01$	$\beta_3 = -1.03$

Numerical simulations confirm the asymptotic expansion.

The Kohn-Vogelius function :

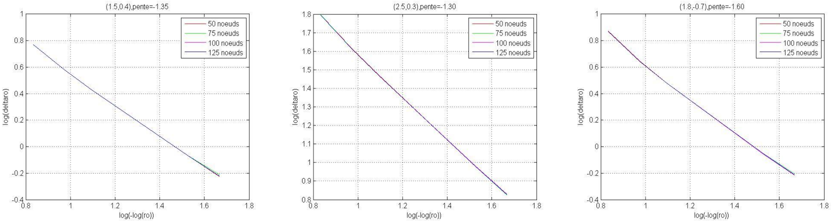


Figure: Variation of $\log(|\Delta_{z_i}(\varepsilon)|)$, $i = 1, ..., 3$ with respect to $\log(-\log(\varepsilon))$.

The perturbation	ω^1	ω^2	ω^3
The slope β_i	$\beta_1 = -1.35$	$\beta_2 = -1.30$	$\beta_3 = -1.60$

Numerical simulations confirm the asymptotic expansion. A slight superconvergence is observed.

One-shot algorithm:

- Compute the topological sensitivity $\delta j(x,y)$, $(x,y) \in \Omega$,
- determine the plasma location by

$$\Omega_p = \{(x,y) \in \Omega;\, \delta j(x,y) \leq (1-\rho)\, g_{min}\}$$

where $g_{min} = \min_{(x,y)\in\Omega} \delta j(x,y)$ and $\rho \in]0,1[$ is a heuristically determined small parameter.

Location of the plasma region from analytic data

Using the Tracking function :

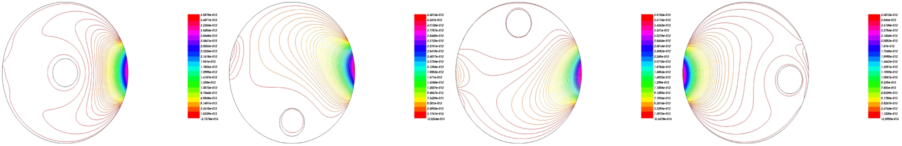


Figure: Various locations of the plasma region

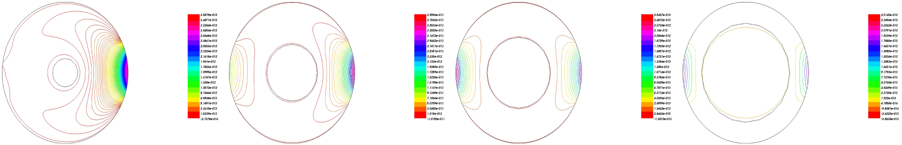


Figure: Various sizes of the plasma region

Flat isovalues $\Longrightarrow$ instability w.r.t. ε.

Location of the plasma region from analytic data

Using the Kohn-Vogelius function :

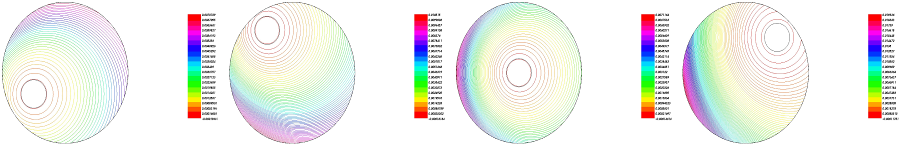


Figure: Various locations of the plasma region

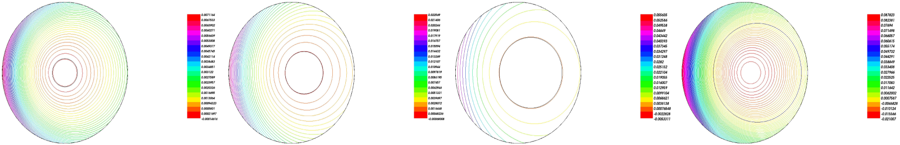


Figure: Various sizes of the plasma region

Sharp isovalues $\Longrightarrow$ stability w.r.t. ε.

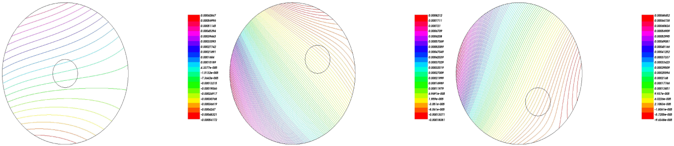


Figure: Reconstruction using the tracking function

Drop the tracking function and use the Kohn-Vogelius one

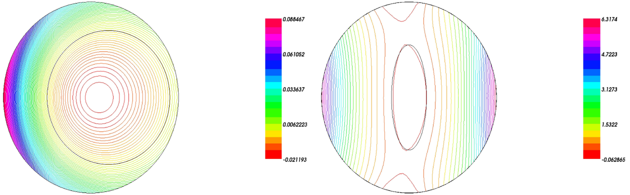


Figure: Reconstruction of a circular and an elliptic shape

the Kohn-Vogelius function efficiently detects both the shape and the location of smooth geometries.

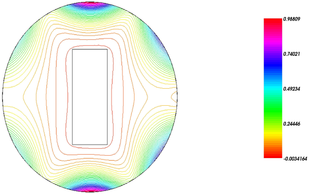


Figure: Reconstruction of shapes with corners

We detect efficiency the location and the shape of a rectangular domain.

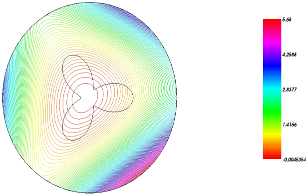


Figure: Reconstruction of a complex geometry

The algorithm locates the region, but fails in reconstructing its shape.

The topological asymptotics provides us with

- A fast algorithm to locate and reconstruct plasma regions in a tokamak
- Its accuracy is good for simple geometries, and quite poor for complex ones
- Might be enough however for real time applications, where

Features :

- No need of a priori information on the location
- One shot algorithm, no iterations
- Once we get the measured data, computations are run on the safe domain only

Prospects for a better accuracy:

- Build an iterative algorithm using the topological asymptotics
- Use the present algorithm as a first guess provider for some shape optimization one ... however more expencive

Thank you for your attention