Numerical methods for a hyperbolic system of balance laws arising in the formation of sandpile on a finite table

Based on Discontinuous flux for Hyperbolic Conservation Laws

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(joint work with Aekta and Adimurthi)

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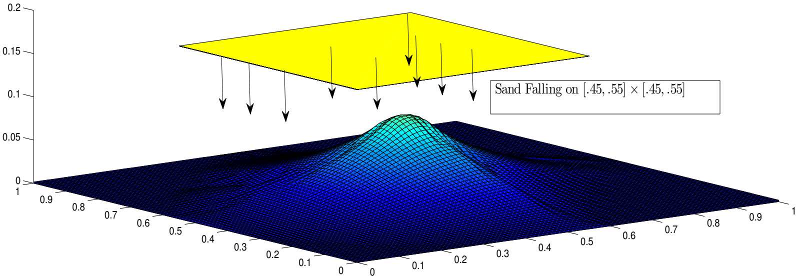
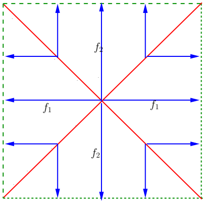
Outline

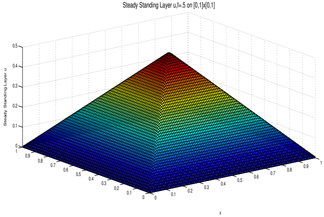
- Hadeler-Kuttler system(HK) for Sandpile Growth
- Application of The Discontinuous Flux to HK system
- Discretization of the system in 1 dimension
- Extension to 2 dimensions
- Open table and Partially Open table problems
- Comparison with the finite difference scheme by Vita and Falcone
- Numerical results

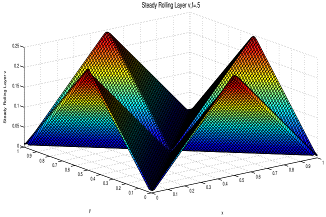


- **(a) Standing layer:** $u(x,t)$ collects the amount of matter that remains at rest
- **(b) Rolling Layer:** $v(x,t)$ represents matter moving down along the surface of the standing layer.
- No existence and uniqueness results known

Open Square: $\Gamma_0 = \partial\Omega$







Top:Singular/Ridge set={ points where distance function is not differentiable}

Bottom: Standing Layer and Rolling Layer

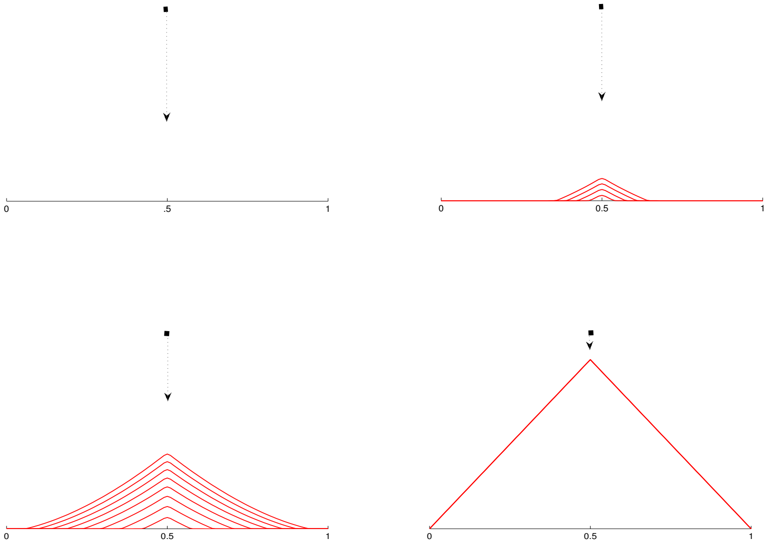
A 2 layer system in 2d;Hadeler-Kuttler Model

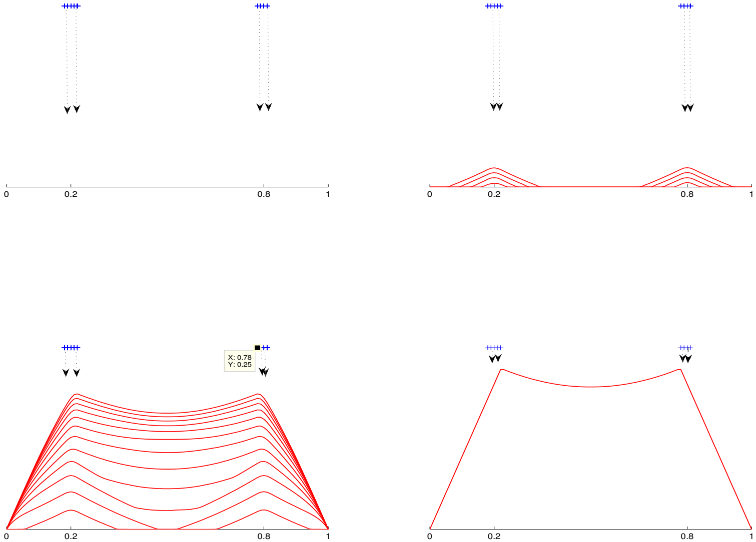
$$v_t - (vu_x)_x - (vu_y)_y = f - (1 - \sqrt{u_x^2 + u_y^2})v, \text{ on } \Omega \times (0, T)$$

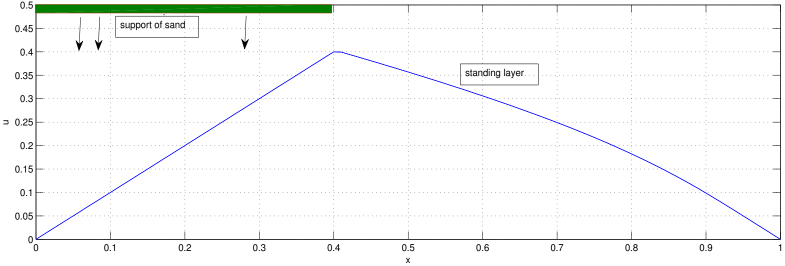
$$u_t = (1 - \sqrt{u_x^2 + u_y^2})v, \text{in } \Omega \times (0, T)$$

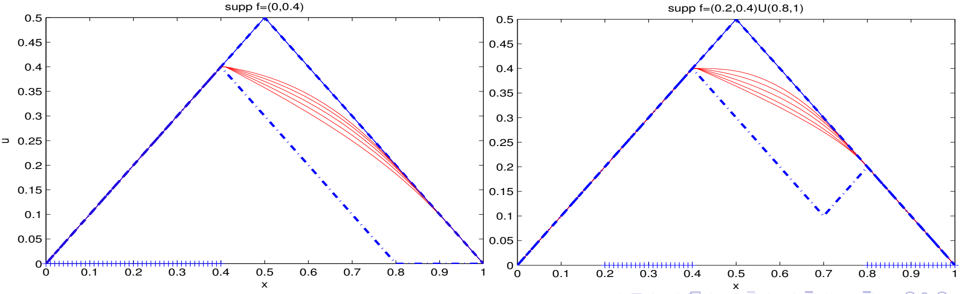
$$\text{with } u_0 = 0, v_0 = 0 \text{ on } \Omega, u = 0 \text{ on } \partial\Omega$$

One dimensional steady state solution




0
0.2
0.8
1
X: 0.78
Y: 0.25


support of sand
standing layer
u
x


supp f=(0,0.4)
supp f=(0.2,0.4)U(0.8,1)
u
x
x

A 2 layer system in 1d

$$v_t - (vu_x)_x = f - (1 - |u_x|)v, \ \text{ on } \ \Omega \times (0, T)$$

$$u_t = (1 - |u_x|)v, \text{in } \ \Omega \times (0, T)$$

$$\text{with } \ u_0 = 0, v_0 = 0 \text{ on } \ \Omega, u = 0 \text{ on } \ \partial\Omega$$

By setting $\alpha = u_x$,last equation becomes

$$\alpha_t + ((|\alpha| - 1)v)_x = 0$$

$$\alpha_t + G(\alpha, v)_x = 0$$

where G is convex in w and $G(-1, v) = G(1, v) = 0$ for all v

Ref:1) Amadori, D., Shen,W. (2009). Global Existence of large BV solutions in a Model of Granular Flow. Communications in Partial Differential Equations, 34(9), 1003-1040.
2) Cannarsa, P., Cardaliaguet, P. (2004). Representation of equilibrium solutions to the table problem of growing sandpiles. Journal of the European Mathematical Society, 6(4), 435-464.

Discontinuous Flux Approach

- The equation for v,

$$v_t - (v u_x)_x = f - (1 - |u_x|)v....$$

$-v u_x$ is a linear function with discontinuous coefficient u_x

AIM: Add source f as a part of the convection term

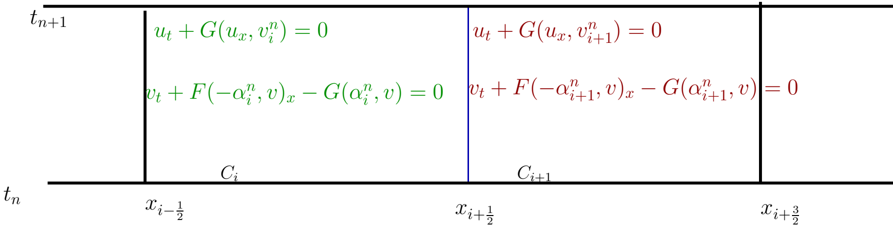
Let

$$g(x) = \int_0^x f(y)\, dy\ , \qquad F(-u_x, v) = -(vu_x - g)$$

. Then equation becomes

$$v_t + F(-u_x, v)_x + (1 - |u_x|)v = 0$$

We use the idea of Discontinuous Flux for Hyperbolic Conservation Laws.



Structure of the Reimann Problem for the system

where

$$\alpha_i^n = (u_{i+\frac{1}{2}}^n - u_{i-\frac{1}{2}}^n)/h$$

$G(u_x, v_i^n) \approx ((-1 + |u_x|)v)_{x_i}^{t_n}$

$F(\alpha_i^n, v) \approx (-u_x v - g)_{x_i+\frac{1}{2}}^{t_n}$ where $g(x) \approx (\int_0^x f(y)\, dy)$

How to define $G(u_x, v_i^n)$ and $F(\alpha_i^n, v)$?

Discretization of the equation α

Discontinuous Godunov Flux for $(|\alpha| - 1)v =$

$$\max((|\max(\alpha_i^n, 0)| - 1)v_i^n, (|\min(\alpha_{i+1}^n, 0)| - 1)v_{i+1}^n)$$

is used.

Ref:(with Adimurthi and Jerome Jaffre) SIAM J. Numer. Anal.42(1):179-208(2004)

Discontinuous Flux

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(F(K(x), u)) = 0, x \in \mathbb{R}, t > 0$$

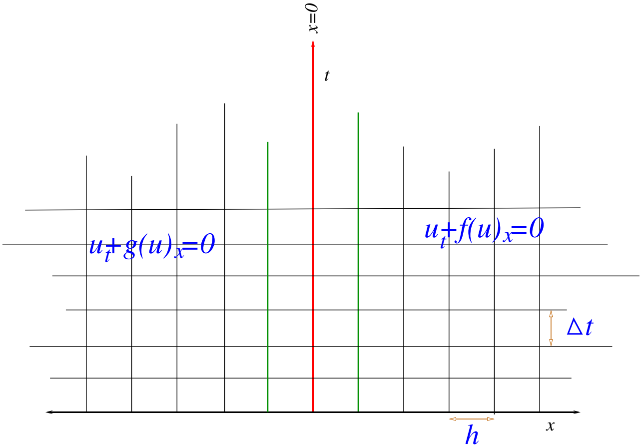
$$u(x, 0) = u_0(x), x \in \mathbb{R}$$

$K(x)$ is a known function discontinuous in x. When $K(x) = H(x)$ is given by

$$F(x, u) = H(x)f(u) + (1 - H(x))g(u)$$

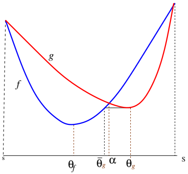
which is a **discontinuous function of x, H is the Heaviside function, where both f and g are smooth and convex type.** **Application: Modeling two phase flow in porous media, sedimentation problem and in traffic flow.**

Godunov Scheme



Compute the solution of **Riemann Problem** by using explicit formula.

Explicit formula for Godunov Flux



- At the interface for convex fluxes f and g:

$$\overline{F}(a,b) = \max\{g(\max(a,\theta_g)), f(\min(\theta_f,b))\}$$

- When $f = g$,f convex,
  $F(a,b) = \max\{f(\max(a,\theta_f)), f(\min(\theta_f,b))\}$
- Scheme is monotone under CFL$=\frac{\Delta t}{h}\sup\max\{f', g'\} \leq 1$
- **Failure of the consistency property**

$$\overline{F}(a,a) \neq g(a) \ \ \textbf{or} \ f(a)$$

Discretization of u-Standing Layer

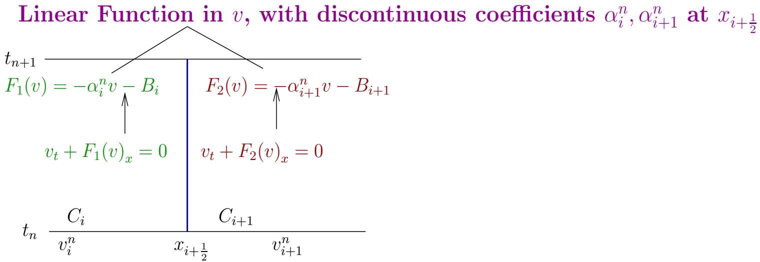
Discontinuous Godunov **(DFLU)** Flux for $(|\alpha| - 1)v$

$$G^n_{i+\frac{1}{2}} = \max((|\max(\alpha^n_i, 0)| - 1)v^n_i, (|\min(\alpha^n_{i+1}, 0)| - 1)v^n_{i+1})$$

and

$$u^{n+1}_{i+\frac{1}{2}} = u^n_{i+\frac{1}{2}} - \Delta t G^n_{i+\frac{1}{2}}$$

Discretization of v-rolling layer



$$v_t - (u_x v - g)_x = -(1 - |u_x|)v$$

Define $B_i = \int_0^{x_i} f(s)\, ds$ then $B_i \leq B_{i+1}$.

We want :

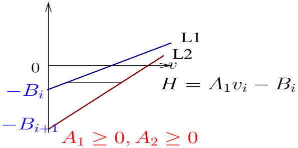
$$v_i^{n+1} = v_i^n - \frac{\Delta t}{h}(H_{i+\frac{1}{2}}^n - H_{i-\frac{1}{2}}^n) + \Delta t v_i^n(|\alpha_i^n| - 1)$$

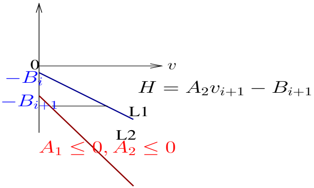
Flux selection $H_{i+\frac{1}{2}}$ in v by solving pb at $x_{i+\frac{1}{2}}$

Existence condition for a weak solution: $F_1(u^-) = F_2(u^+)$

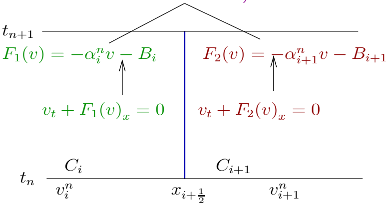
Uniqueness of weak solution: $F_1'(u^-) < 0, F_2'(u^+) > 0$ not allowed

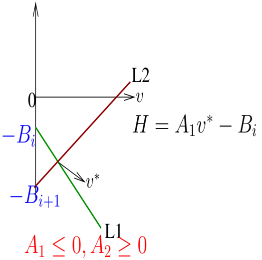
$A_1 = -\alpha_i^n, A_2 = -\alpha_{i+1}^n, L_1 = A_1 v - B_i, L_2 = A_2 v - B_{i+1}$

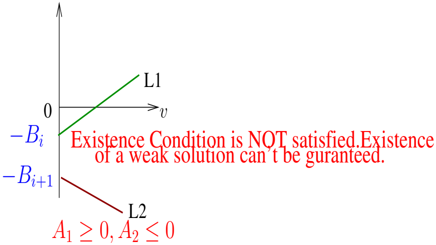




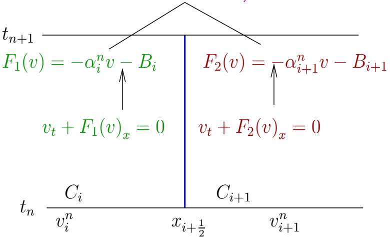
Linear Function in v, with discontinuous coefficients $\alpha_i^n, \alpha_{i+1}^n$ at $x_{i+\frac{1}{2}}$



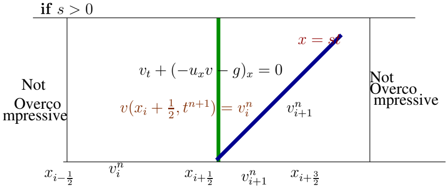


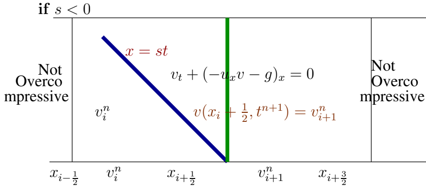


Linear Function in v, with discontinuous coefficients $\alpha_i^n, \alpha_{i+1}^n$ at $x_{i+\frac{1}{2}}$



$$Case: A_1 > 0, A_2 < 0$$





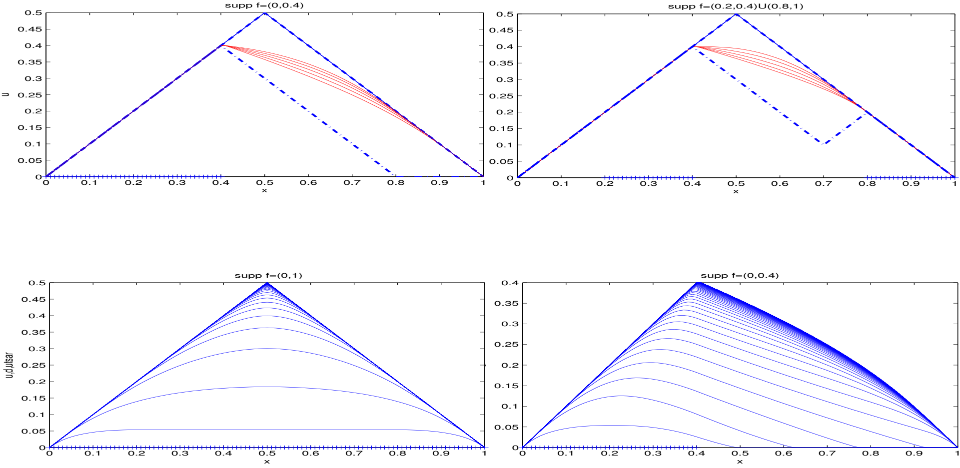
$$s = \frac{(-\alpha_i^n v_i^n - B_i) - (-\alpha_{i+1}^n v_{i+1}^n - B_{i+1})}{v_i^n - v_{i+1}^n} \begin{cases} > 0 & \text{if } v_i^n \geq v_{i+1}^n \\ < 0 & \text{else} \end{cases},$$

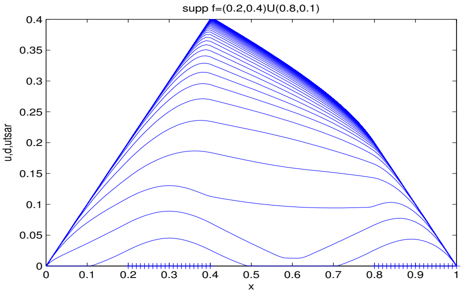
since $B_i \leq B_{i+1}$

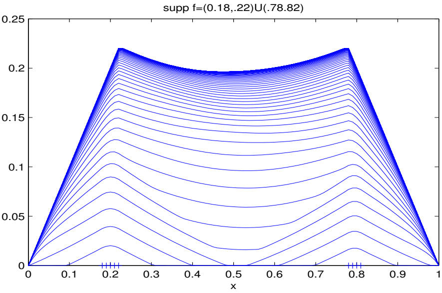
$$\therefore H_{i+\frac{1}{2}}^n = \begin{cases} -\alpha_i^n v_i^n - B_i & \text{if} \quad v_i^n \geq v_{i+1}^n \\ -\alpha_{i+1}^n v_{i+1}^n - B_{i+1} & \text{otherwise} \end{cases}$$

Numerical Results in 1d

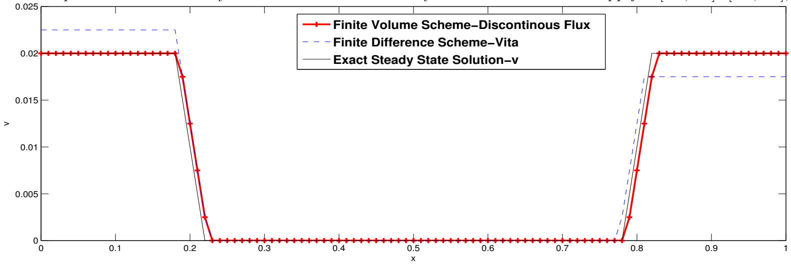
$$0 \le f_1 \le f_2 \ \text{ in } D_f \Rightarrow u_1 \le u_2 \le d \ \text{ in}[0,1].$$







Comparision of exact Steady State Solution v acheieved by both the schemes with supp $f = [.18, .22]U[.78, .82], f = .5$



Properties of the scheme

- This scheme is monotone with an almost uniform convergence rate of 1.
- Physical properties are preserved: $v^n \geq 0, |\alpha^n| \leq 1, u^{n+1} \geq u^n$ under the CFL condition:

$$\boldsymbol{\lambda \leq 1 - \Delta t},$$

and

$$\boldsymbol{\lambda \sup_i v_i^n \leq 1}.$$

- It is well-balanced for piecewise constant source terms in the case when
  - a unique equilibrium (u_f, v_f) exists.
  - otherwise, under the additional assumption that $\alpha_{J+1} = 0, \alpha_i \leq 0$ where $f(x) = 0\ \forall x > x_{J+\frac{1}{2}}$.

Well-Balanced Scheme

M	L^∞ error		L^1 error	
	(FV)	**S**(**T**)	**(FV)**	**S**(**T**)
50	4.85×10^{-16}	.0050	9.60×10^{-17}	7.2×10^{-4}
100	9.85×10^{-16}	.0025	1.008×10^{-16}	1.8×10^{-4}
200	2.31×10^{-15}	.0012	2.08×10^{-16}	4.5×10^{-5}
400	3.95×10^{-15}	6.18×10^{-4}	9.05×10^{-16}	1.12×10^{-5}

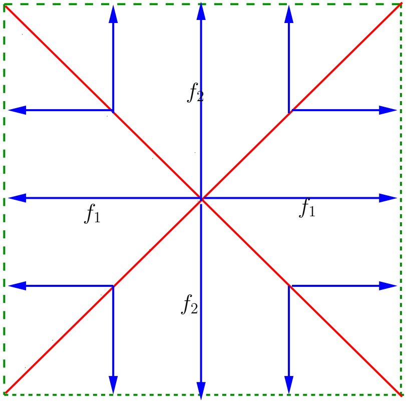
Table: $\lambda = .9$, $T = \Delta t$, double precision error in v_h computed by **(FV)** scheme and **(S)** and **(T)** schemes.

Discretization in 2d

AIM:As in 1 dimension,include f with the convection term $(-vu_x)_x - (vu_y)_y$.

DIFFICULTY:How to distribute the $f(x, y)$ **properly** in both the dimensions!!

$$v_t + (-vu_x)_x - (vu_y)_y = f(x, y)$$



f_2
f_1
f_1
f_2

Approx. of $u_t + (|\nabla u| - 1)v = 0$, Extension of **DFLU** flux

Extend

$$G^n_{i+\frac{1}{2}} = \max(|\max(\alpha^n_i, 0)| - 1)v^n_i, |\min(\alpha^n_{i+1}, 0)| - 1)v^n_{i+1})$$

$$(u_x)^2 \approx G^{n,x}_{i+\frac{1}{2},k} = \max(|\max(\alpha^n_{i,k}, 0)|, |\min(\alpha^n_{i+1,k}, 0)|)^2$$

and

$$(u_y)^2 \approx G^{n,y}_{i,k+\frac{1}{2}} = \max(|\max(\beta^n_{i,k}, 0)|, |\min(\beta^n_{i,k+1,}, 0)|)^2$$

Define

$$W^{n,x}_{i,k+\frac{1}{2}} = (\sqrt{(|\max(\alpha^n_{i,k}, 0)|)^2 + G^{n,y}_{i,k+\frac{1}{2}}} - 1)v^n_{i,k},$$

$$W^{n,x}_{i+1,k+\frac{1}{2}} = (\sqrt{|\min(\alpha^n_{i+1,k}, 0)|^2 + G^{n,y}_{i,k+\frac{1}{2}}} - 1)v^n_{i+1,k},$$

$$W^{n,y}_{i+\frac{1}{2},k} = (\sqrt{|\max(\beta^n_{i,k}, 0)|^2 + G^{n,x}_{i+\frac{1}{2},k}} - 1)v^n_{i,k} \text{ and}$$

$$W^{n,y}_{i+\frac{1}{2},k+1} = (\sqrt{|\min(\beta^n_{i,k+1}, 0)|^2 + G^{n,x}_{i+\frac{1}{2},k}} - 1)v^n_{i,k+1}$$

Define the numerical fluxes in the $x-$ direction by

$$Gx^n_{i+\frac{1}{2},k+\frac{1}{2}} = \max(W^{n,x}_{i,k+\frac{1}{2}}, W^{n,x}_{i+1,k+\frac{1}{2}})$$

and in the $y-$ direction

$$Gy^n_{i+\frac{1}{2},k+\frac{1}{2}} = \max(W^{n,y}_{i+\frac{1}{2},k}, W^{n,y}_{i+\frac{1}{2},k+1})$$

respectively. Now, define

$$G^n_{i+\frac{1}{2},k+\frac{1}{2}} = \max(Gx^{n,x}_{i+\frac{1}{2},k}, Gy^{n,y}_{i,k+\frac{1}{2}})$$

- Consistency with the upwind flux by Rouy and Tourin the case when v is continuous.
- Consistency with one dimensional **DFLU** flux.

$$u^{n+1}_{i+\frac{1}{2},k+\frac{1}{2}} = u^n_{i+\frac{1}{2},k+\frac{1}{2}} - \Delta t G^n_{i+\frac{1}{2},k+\frac{1}{2}}$$

Numerical Scheme for Equation in v

$$v_t + (-vu_x)_x + (-vu_y)_y = f + ...$$

Aim:

$$v_t + (-vu_x - g^x)_x + (-vu_y - g^y)_y = ...$$

We want to solve

$$g^x_x + g^y_y = f$$

Define $g^x(x,y) = \int_0^x f_1(z,y)dz \ \ \forall y \in [0,1]$

$$g^y(x,y) = \int_0^y f_2(x,z)dz \ \ \forall x \in [0,1]$$

where f_1, f_2 denotes the distribution of source in $x, y-$ direction respectively.

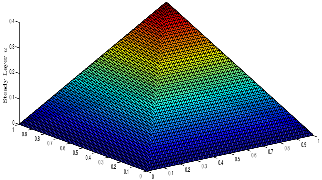
Calculate H^x and H^y as in 1 dimension by solving Riemann Problem for v in each direction.

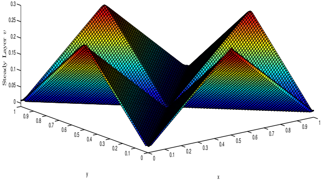
The equation(9) can be approximated by:

$$v_{i,k}^{n+1} = v_{i,k}^{n} - \lambda(F^{n}_{x\,i+\frac{1}{2},k+\frac{1}{2}} - F^{n}_{x\,i-\frac{1}{2},k+\frac{1}{2}}) - \lambda(F^{n}_{y\,i+\frac{1}{2},k+\frac{1}{2}} - F^{n}_{y\,i+\frac{1}{2},k-\frac{1}{2}})$$
$$+ \Delta t v_{i,k}^{n}(\sqrt{(\alpha_{i,k}^{n})^2 + (\beta_{i,k}^{n})^2} - 1)$$

Numerical Experiments in 2d

$f = .5$ on $[0, 1] \times [0, 1]$





CFL comparisons with the FDM by Falcone and Vita

Discontinous Flux Scheme

N	f=1	f=2	f=3
201	.995	.995	.67
401	.9975	.9975	.6683
601	.9983	.9983	.6678
801	.9988	.9988	.6675

Finte Difference Scheme[FV06]

N	f=1	f=2	f=3
201	.6	.52	.46
401	.552	.476	.424
601	.528	.45	.41
801	.5	.43	.39

Comparision in 1 dimension

Discontinous Flux Scheme

N	f=.5	$f = max(0, sin(2\pi x) + cos(2\pi y) - sin(2\pi xy)$
51	.9	.9
101	.8979	.7354
201	.8058	.6448

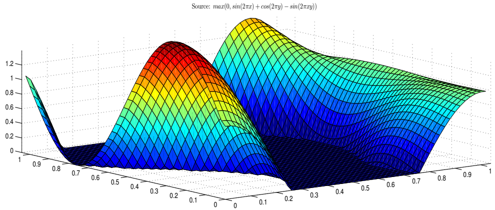
Finte Difference Scheme[FV06]

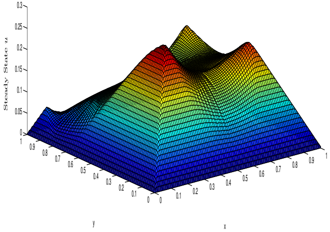
N	f=.5	$f = max(0, sin(2\pi x) + cos(2\pi y) - sin(2\pi xy)$
51	.7	.6
101	.6	.5
201	.4	.4

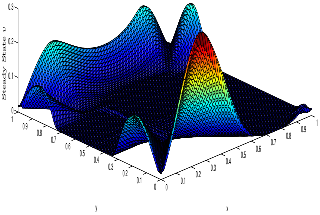
Comparision in 2 dimension

- Since $\lambda * \max_{[0,1]} v \leq 1$ and $||v||_\infty \leq C||f||_\infty$
  CFL decreases as f gets bigger,
  but for our scheme the decrease is much slower.

$$f = \max(0, sin(2\pi x) + cos(2\pi y) - sin(2\pi xy))$$

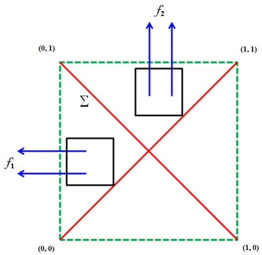






Open Square:

$D_f = [0.1, 0.3] \times [0.5, 0.7] \cup [0.5, 0.7] \times [0.7, 0.9], f = .5$



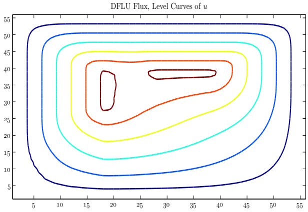
$\Gamma_0 = \partial\Omega$ [- - -],

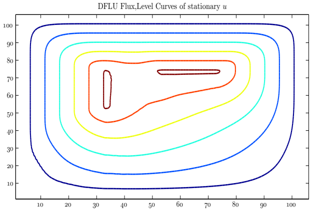
Transport rays [$\rightarrow$],

Σ [—],

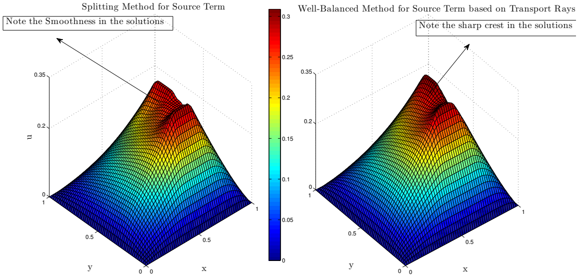
$h = 1/55, 1/105$,

$\lambda = .7$, DFLU flux

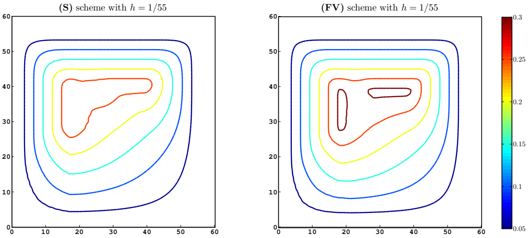




Source term distribution crucial for capturing harder singularities like crests



$f = .5, h = 1/55$



Crest and level curves

Boundary Conditions for the table problem

Let $\partial\Omega = \Gamma_0 \cup \Gamma_w$, Γ_0, a non-empty closed subset of Ω.

1. Open Table Problem($\Gamma_0 = \partial\Omega$): The sand can fall from the table from each point of the boundary and nothing can deposit on $\partial\Omega$.
2. Partially Open Table Problem($\emptyset \neq \Gamma_0 \subset \partial\Omega$ and is closed): The sand can fall from the table from each point of Γ_0 and nothing can deposit on Γ_0. No sand falls from the table from any point of Γ_w.

Boundary with wall means:

$$v\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_w$$

The Singular Set of the Distance Function from Γ_0

- Let $d_{\Gamma_0} : \overline{\Omega} \to \mathbb{R}$ be the distance function from the set Γ_0 defined by

$$d_{\Gamma_0}(x) = \min_{y \in \Gamma_0} |y - x|, \;\; x \in \Omega$$

- Let $\Pi_{\Gamma_0} : \overline{\Omega} \to \Gamma_0$ be the projection of x to the Γ_0 i.e.

$$\Pi_{\Gamma_0}(x) = \{y \in \Gamma_0 : d_{\Gamma_0}(x) = |y - x|\}$$

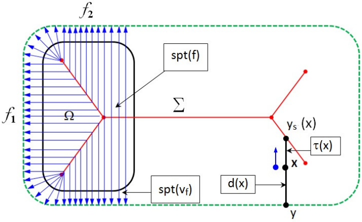
- *Singular set/Ridge* of d_{Γ_0} and is denoted by Σ; equivalently

$$\Sigma = \{x \in \overline{\Omega}, l(x) = d_{\Gamma_0}(x)\}$$

where $l(x)$ is the length of the tranport rays.

Singular Set and Transport Rays

- Pick any $x \in \Omega$ which is not a boundary point and also not in the singular set Σ. Find the point $y = y(x)$ on the boundary which has minimal distance from x. Now draw the straight line joining y and x and beyond till it first touches the singular set Σ in $y_s(x)$. This is the transport ray through x.



- These are the lines along which the rolling layer flows from the source down to the edge of the table.

$\Gamma_0 = \{0 < x < 1, y = 0\}$

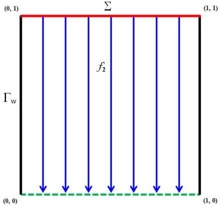
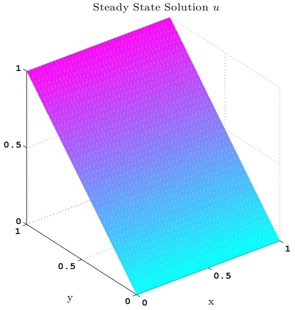
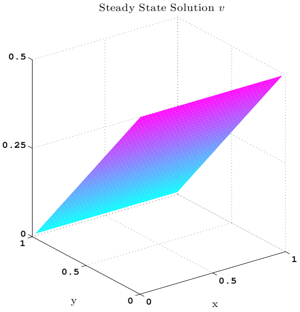
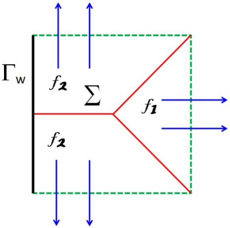


Figure: Γ_0[- - -], Transport rays[→], Σ[—]

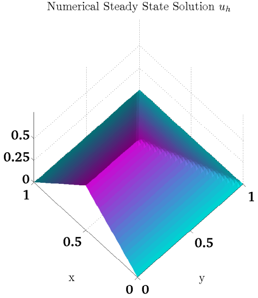


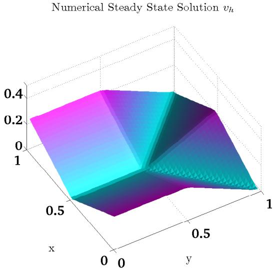


$\Gamma_w = \{0 < y < 1, x = 0\}$

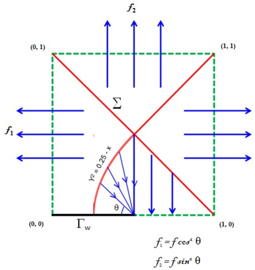


Γ_0 [- - -],
Transport rays [→],
Σ [—]

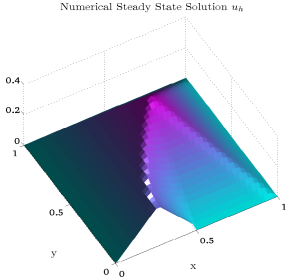


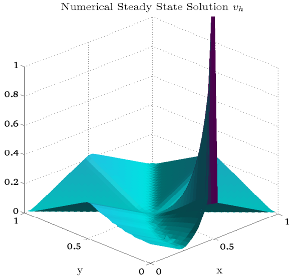


Only Half wall closed ,$\Gamma_w = \{0 < x < .5, y = 0\}$

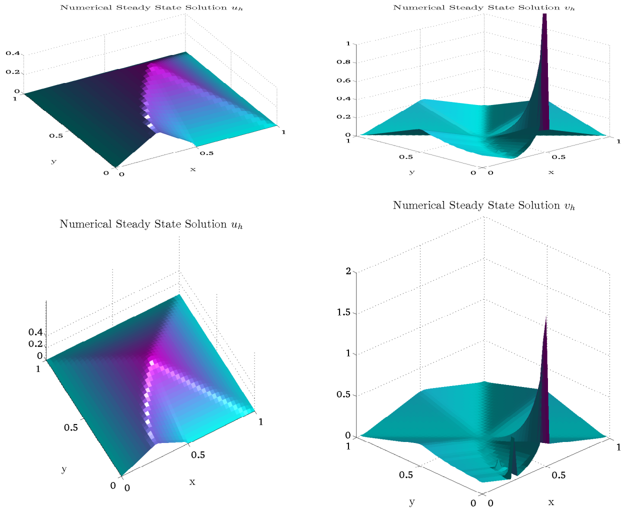


Γ_0 [- - -],
Transport rays [$\rightarrow$],
Σ [—]



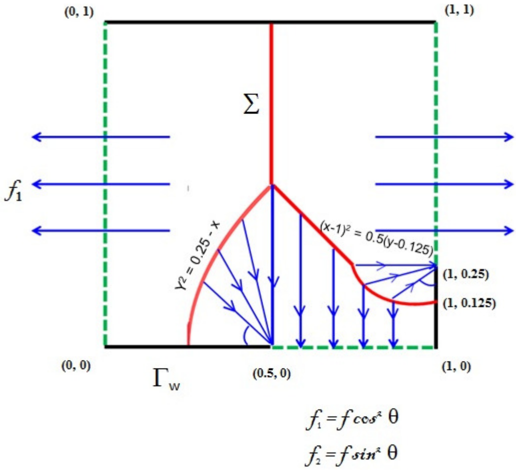


Comparision with standard splitting for source term



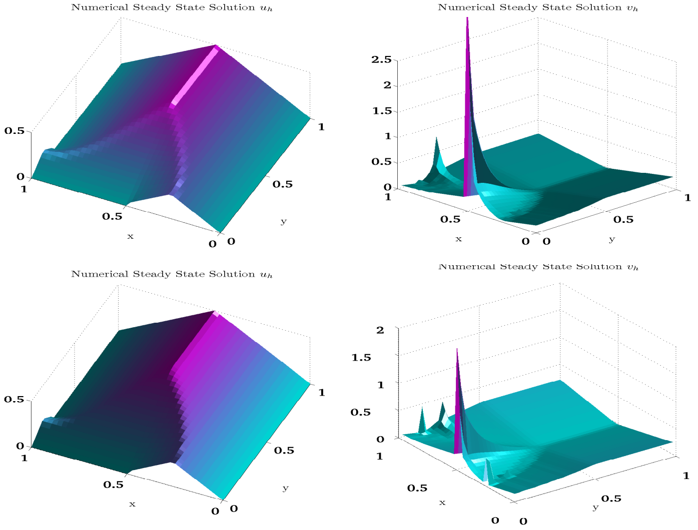
Top: source term Inside, Bottom: splitting method.

$\Gamma_w = \{0 < x < .5, y = 0\} \cup \{0 < y < .25, x = 1\} \cup \{0 < x < 1, y = 1\}$



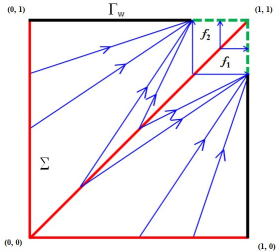
Γ_0 [- - -],
Transport rays [→],
Σ[—]

Comparision with standard splitting for source term

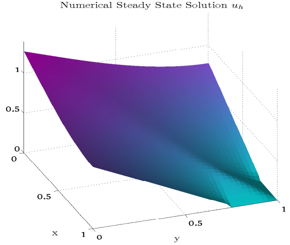


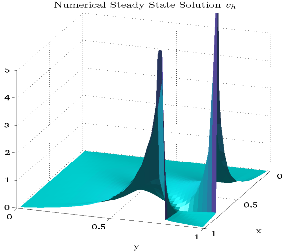
Top: source term inside, Bottom: splitting method.

$\Gamma = \{.75 < x < 1, y = 1\} \cup \{.75 < y < 1, x = 1\}$



Γ_0 [- - -],
Transport rays [→],
Σ[—]





$\Gamma = \{0 < x < .5, y = 0\}$

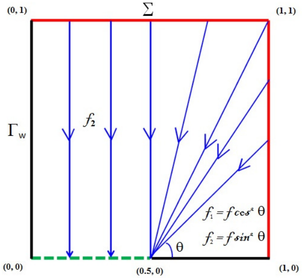
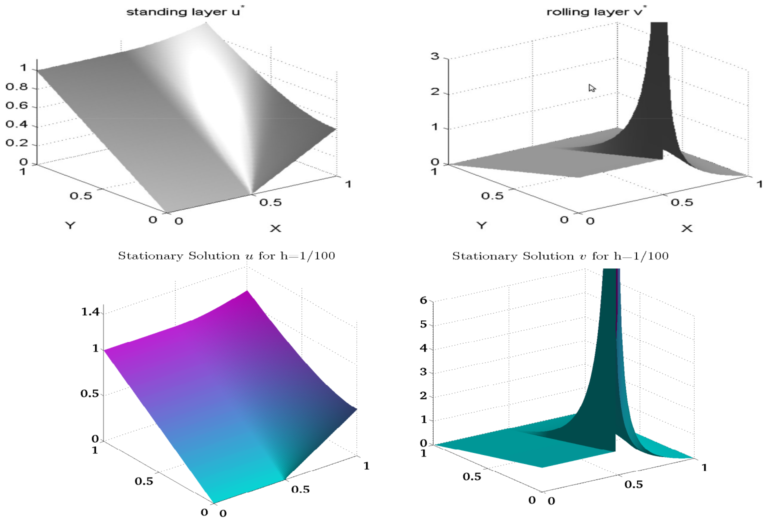


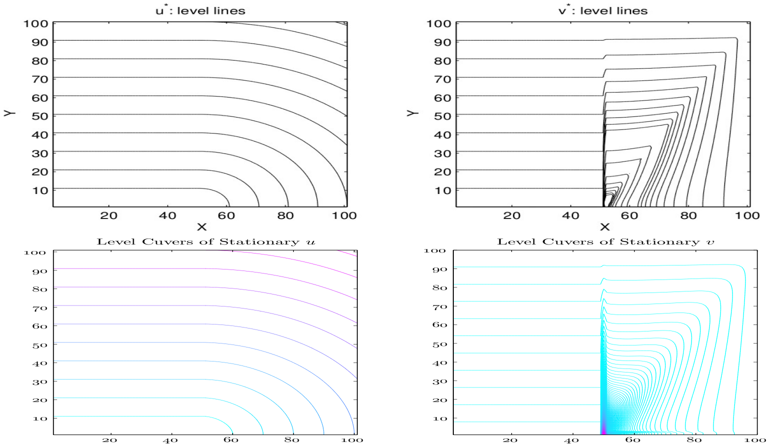
Figure: Γ_0[- - -],Transport rays[→],Σ[—]

$\Gamma = \{0 < x < .5, y = 0\}$

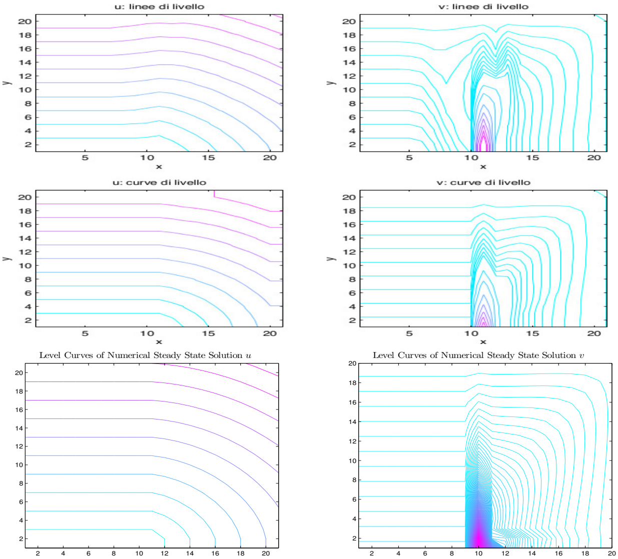


Top:Exact,Below:Numerical

$\Gamma = \{0 < x < .5, y = 0\}$

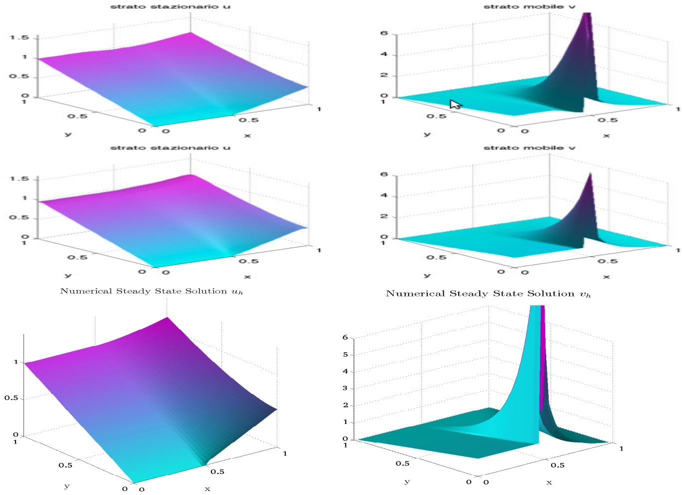


Top:Exact,Below:Numerical



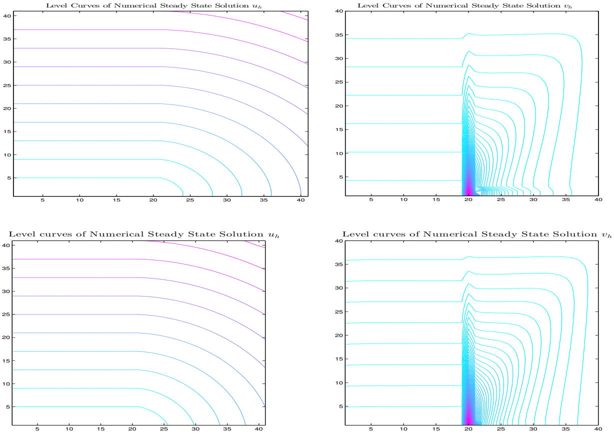
Top:Finite Difference,Middle:Semi Langrangian,

Below: Discontinous Flux



Top:Finite Difference,Middle:Semi Langrangian,

Below: Discontinous Flux



Top:Discontinous Flux,Below:Discontinous Flux with extension

- Developed a Well-Balanced First Order Finite Volume Scheme.
- Developed a scheme for Hamiltonians in Multi-Dimensions using Discontinuous Flux for hyperbolic conservation laws.
- Developed a New Technique for Source Distribution in multi dimensions for non-additive discontinuous sources.

This work is dedicated to Prof.Jerome Jaffree on the occassion of his 65th Birthday

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