Total Differential interpolation
of Two-phase Data
for Three-phase Compressible Flows

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Summary

• Three-phase compressible equations
• Choice of pressure unknown: water, oil or gas versus global?
• TD-condition $\Leftrightarrow$ existence of a global capillary function $P_{cg}$
• Interpolation of TD-three-phase data from two-phase data
• Numerical results
• Conclusions
Three-phase Compressible equations

- Classical resolution (1=water, 2=oil, 3=gas):
  
  solve for the oil pressure $P_2$ the “pressure equation”:

  \[
  \frac{\partial}{\partial t} \left\{ \phi \sum_{j=1}^{3} B_j S_j \right\} + \nabla \cdot q = 0,
  \]

  where $q$ is the global volumetric flow vector:

  \[
  q = \sum_{j=1}^{3} \varphi_j = -K d \left\{ \nabla P_2 + f_1 \nabla P_{c1}^{12} + f_3 \nabla P_{c3}^{32} - \rho g \nabla Z \right\}
  \]

  \[
  \left\{ 
  \begin{array}{l}
  d(s_1, s_3, p_2) = \sum_{j=1}^{3} kr_j d_j = \text{global mobility}, \\
  f_j(s_1, s_3, p_2) = kr_j d_j / \lambda = j^{th} \text{ fractional flow}, \sum_{j=1}^{3} f_j = 1, \\
  \rho(s_1, s_3, p_2) = \sum_{j=1}^{3} f_j \rho_j = \text{global density}.
  \end{array}
  \right.
  \]
Three-phase Compressible equations

- Classical resolution (1=water, 2=oil, 3=gas):

  solve for the oil pressure $P_2$ the “pressure equation”:

  $$\frac{\partial}{\partial t}\{\phi \sum_{j=1}^{3} B_j S_j\} + \nabla \cdot q = 0,$$

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  $$q = \sum_{j=1}^{3} \varphi_j = -Kd\{\nabla P_2 + f_1 \nabla P_c^{12} + f_3 \nabla P_c^{32} - \rho g \nabla Z\}$$

  \[
  \begin{aligned}
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  f_j(s_1, s_3, p_2) &= kr_j d_j / \lambda = j^{th} \textit{fractional flow} , \sum_{j=1}^{3} f_j = 1, \\
  \rho(s_1, s_3, p_2) &= \sum_{j=1}^{3} f_j \rho_j = \textit{global density}.
  \end{aligned}
  \]

  Is oil pressure a better unknown than water or gas pressure?


Pressure unknown: water, oil or gas? (1)

- Capillary pressures (Van Genuchten):

  they depend only on the wetting phase saturation!

\[
\text{top: } P_{c}^{32} = P_{gas} - P_{oil}
\]

\[
\text{bottom: } P_{c}^{12} = P_{water} - P_{oil}
\]
Does there exist a pressure field \((x, t) \rightarrow P\) such that:

Happy retirement Jean and Jérôme!
Let us have a dream...

- Does there exist a pressure field \((x, t) \mapsto P\) such that:

  \[
P \text{ is smooth}, \quad P_{\text{water}} \leq P \leq P_{\text{oil}}
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Does there exist a pressure field \((x, t) \mapsto P\) such that:

- \(P\) is smooth
- \(P_{\text{water}} \leq P \leq P_{\text{oil}}\)

and \(P\) governs the global volumetric flow vector \(q\):

\[
q = -Kd(s_1, s_3, P)\{\nabla P - \rho g \nabla Z\}
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  and \(P\) governs the global volumetric flow vector \(q\):
  
  \[ q = -Kd(s_1, s_3, p)\left\{ (1 - \partial P_{cg}/\partial P) \nabla P - \rho g \nabla Z \right\} \]

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There exists a Global Capillary Pressure function $P_{cg}$ s. t.:

\[
\begin{align*}
\frac{\partial P_{cg}}{\partial S_1}(s_1, s_3, p) &= f_1(s_1, s_3, p - P_{cg}(s_1, s_3, p)) \frac{dP_c^{12}}{dS_1}(s_1), \\
\frac{\partial P_{cg}}{\partial S_3}(s_1, s_3, p) &= f_3(s_1, s_3, p - P_{cg}(s_1, s_3, p)) \frac{dP_c^{32}}{dS_3}(s_3),
\end{align*}
\]

for $P_{\text{min}} \leq p \leq P_{\text{max}}$, $s = (s_1, s_3) \in \mathbb{T}$.

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Total Differential Condition (GC, Applicable Analysis 2009)

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  \end{align*}
  \]

  for $P_{min} \leq p \leq P_{max}$, $s = (s_1, s_3) \in T$.

- **Condition** $p_1 \leq p \leq p_3$ is satisfied if :

  $P_{cg}(1, 0, p) = 0$
Total Differential Condition (GC, Applicable Analysis 2009)

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\frac{\partial P_{cg}}{\partial S_3}(s_1, s_3, p) &= f_3(s_1, s_3, p - P_{cg}(s_1, s_3, p)) \frac{dP_{32}^c}{ds_3}(s_3),
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- Condition $p_1 \leq p \leq p_3$ is satisfied if :

$P_{cg}(1, 0, p) = 0$

- When $P_{cg}$ exists, necessarily

$P_{cg}(s) = 0 + \int_0^1 \nabla_s P_{cg}(\mathcal{C}(t)) \cdot \mathcal{C}'(t) dt$

is independant of $\mathcal{C}$ !

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is independant of $C$!
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\frac{\partial P_{cg}}{\partial S_1}(s_1, s_3, p) &= f_1(s_1, s_3, p - P_{cg}(s_1, s_3, p)) \frac{dP_{c12}^{12}}{dS_1}(s_1), \\
\frac{\partial P_{cg}}{\partial S_3}(s_1, s_3, p) &= f_3(s_1, s_3, p - P_{cg}(s_1, s_3, p)) \frac{dP_{c32}^{32}}{dS_3}(s_3),
\end{align*}
\]

for $P_{min} \leq p \leq P_{max}$, $s = (s_1, s_3) \in \mathbb{T}$.

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\frac{\partial P_{cg}}{\partial S_3}(s_1, s_3, p) &= f_3(s_1, s_3, p - P_{cg}(s_1, s_3, p)) \frac{dP_{c32}}{ds_3}(s_3),
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for $P_{min} \leq p \leq P_{max}$, $s = (s_1, s_3) \in \mathbb{T}$.

Condition $p_1 \leq p \leq p_3$ is satisfied if :

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\[ P_{cg}(s) = 0 + \int_0^1 \nabla_s P_{cg}(C(t)) \cdot C'(t) dt \]

is independent of $C$.

\[ \downarrow \]

TD-condition on $kr_1, kr_2, kr_3, P_{c12}, P_{c32}$

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Global pressure formulation:

- When $P_{cg}$ exists, the Global Pressure $P$ is defined by:

$$P - P_2 = P_{cg}(s, P)$$

- $P$ governs the global flow $q$:

$$q = -Kd(s_1, s_3, P)\{(1 - \partial P_{cg}/\partial P)\nabla P - \rho g \nabla Z\}$$
Global pressure formulation:

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$$q = -Kd(s_1, s_3, P)\{ (1 - \frac{\partial P_{cg}}{\partial P})\nabla P - \rho g \nabla Z \}$$

- For a single compressible fluid (gas), one has:

$$0 < \frac{\partial P_{cg}}{\partial P} \leq 1 \quad \text{over} \ T$$

- Numerical results show that:

$$0 < \frac{\partial P_{cg}}{\partial P} \leq 10^{-4} \quad \text{hence:} \quad 1 - \frac{\partial P_{cg}}{\partial P} \simeq 1$$
Global Capillary Pressure function $P_{cg} = P - P_2$
Degrees of freedom for TD relative permeabilities:

At each global pressure level $p$

- one can choose freely only two functions over $\mathbb{T}$:
  
  $s \in \mathbb{T} \mapsto P_{cg}(s, p)$  \hspace{0.5cm} \text{global capillary pressure function}

  $s \in \mathbb{T} \mapsto d(s, p)$  \hspace{0.5cm} \text{global mobility function}
Degrees of freedom for TD relative permeabilities:

At each global pressure level $p$

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  \begin{align*}
  s \in \mathbb{T} & \mapsto P_{cg}(s, p) \quad \text{global capillary pressure function} \\
  s \in \mathbb{T} & \mapsto d(s, p) \quad \text{global mobility function}
  \end{align*}

- then associated TD fractional flows are given by:
  \[ f_j(s, p) = \frac{\partial P_{cg}/\partial s_j(s, p)}{dP_c^j/\partial s_j(s_j)} , \quad j = 1, 3 \]
Degrees of freedom for TD relative permeabilities:

At each global pressure level $p$

- one can choose freely only two functions over $T$:
  
  $s \in T \sim P_{cg}(s,p)$ \quad \text{global capillary pressure function}
  
  $s \in T \sim d(s,p)$ \quad \text{global mobility function}

- then associated TD fractional flows are given by:
  
  $f_j(s,p) = \frac{\partial P_{cg}/\partial s_j(s,p)}{dP_c^{j2}/ds_j(s_j)}$, \quad j = 1, 3

- and associated TD relative permeabilities by:
  
  \[
  \begin{cases}
  k_{rj}(s,p) = f_j(s,p) d(s,p) / d_j(p - P_{cg}(s,p) + P_c^{j2}(s_j)) \quad j = 1, 3, \\
  k_{r2}(s,p) = (1 - f_1(s,p) - f_3(s,p)) d(s,p) / d_2(p - P_{cg}(s,p)) ,
  \end{cases}
  \]
Construction of TD-relative permeabilities

Available experimental data:
- for each fluid $j = 1, 2, 3$:
  - mobilities $d_j(p_j) = B_j(p_j)/\mu_j(p_j)$,
Construction of TD-relative permeabilities

Available experimental data:
- for each fluid $j = 1, 2, 3$:
  - mobilities $d_j(p_j) = B_j(p_j)/\mu_j(p_j)$,
- for each pair of fluids $i, j$:

\[
\begin{align*}
&\text{water} \quad \text{gas} \quad \text{oil} \\
&\text{water - gas data} \quad \text{gas - oil data} \quad \text{water - oil data}
\end{align*}
\]
Construction of **TD-relative permeabilities**

**Available experimental data:**

- for each fluid $j = 1, 2, 3$:
  - mobilities $d_j(p_j) = B_j(p_j)/\mu_j(p_j)$,
- for each pair of fluids $i, j$:
  - relative permeabilities $k_{r_{i,j}^i}^i, k_{r_{j,i}^j}^j$ (Mualem-Van Genuchten)
Construction of TD-relative permeabilities

Available experimental data:

• for each fluid $j = 1, 2, 3$:
  mobilities $d_j(p_j) = B_j(p_j) / \mu_j(p_j)$,

• for each pair of fluids $i, j$:
  relative permeabilities $k_{r_i^j}$, $k_{r_j^i}$ (Mualem-Van Genuchten)

• little known on the $k_r$’s inside $\mathbb{T}$!
Construction of TD-relative permeabilities

Available experimental data:

- for each fluid $j = 1, 2, 3$:
  
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  relative permeabilities $k_{r_{i,j}}^i, k_{r_{j,i}}^j$ (Mualem-Van Genuchten)

- little known on the $k_r$’s inside $\mathbb{T}$!

Question:

can TD-three-phase relative permeabilities on $\mathbb{T}$ honor two-phase data on $\partial\mathbb{T}$?
Honoring two-phase data: 1 Determine $P_{cg}$ on $\partial \mathbb{T}$

- Let $p$ be a given global pressure level.
- Recall: on any curve $C : [0, 1] \rightarrow \mathbb{T}$, $P_{cg}(C(t))$ satisfies:

\[
(1) \quad \frac{dP_{cg}}{dt} = f_1(C, p - P_{cg}) \frac{dP^{12}_{c}}{ds_1}(C_1) C'_1 + f_3(C, p - P_{cg}) \frac{dP^{32}_{c}}{ds_3}(C_3) C'_3,
\]

- Use (1) to determine of $P_{cg}$ on $\partial \mathbb{T}$:
Honoring two-phase data: 1 Determine $P_{cg}$ on $\partial \mathbb{T}$

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- Use (1) to determine of $P_{cg}$ on $\partial \mathbb{T}$:
  - on the water-gas side $\Rightarrow P_{cg}^{13}$

\[ P_{cg} = 0 \]

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Honoring two-phase data: 1 Determine $P_{cg}$ on $\partial \mathbb{T}$

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- Recall: on any curve $C : [0, 1] \rightarrow \mathbb{T}$, $P_{cg}(C(t))$ satisfies:

$$
(1) \quad \frac{dP_{cg}}{dt} = f_1(C, p - P_{cg}) \frac{dP_c^{12}}{ds_1}(C_1) C'_1 + f_3(C, p - P_{cg}) \frac{dP_c^{32}}{ds_3}(C_3) C'_3,
$$

- Use (1) to determine of $P_{cg}$ on $\partial \mathbb{T}$:
  - on the water-gas side $\quad \Rightarrow \quad P_{cg}^{13}$
  - on the water-oil side $\quad \Rightarrow \quad P_{cg}^{12}$

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Honoring two-phase data : 1 Determine $P_{cg}$ on $\partial \mathbb{T}$

- Let $p$ be a given global pressure level.
- Recall: on any curve $C : [0, 1] \rightarrow \mathbb{T}$, $P_{cg}(C(t))$ satisfies:

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(1) \quad \frac{dP_{cg}}{dt} = f_1(C, p - P_{cg}) \left( \frac{dP_{c12}^{12}}{ds_1} C_1' + f_3(C, p - P_{cg}) \frac{dP_{c32}^{32}}{ds_3} C_3' \right),
$$

- Use (1) to determine of $P_{cg}$ on $\partial \mathbb{T}$:
  - on the water-gas side $\Rightarrow P_{c13}^{13}$
  - on the water-oil side $\Rightarrow P_{c12}^{12}$
  - on the gas-oil side $\Rightarrow P_{c23}^{23}$

$P_{cg}^{32}(0) = P_{cg}^{12}(1)$
Honoring two-phase data: 1 Determine $P_{cg}$ on $\partial \mathbb{T}$

- Let $p$ be a given global pressure level.
- Recall: on any curve $C : [0, 1] \rightarrow \mathbb{T}$, $P_{cg}(C(t))$ satisfies:

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(1) \quad \frac{dP_{cg}}{dt} = f_1(C, p - P_{cg}) \frac{dP_{12}^c}{ds_1}(C_1) C_1' + f_3(C, p - P_{cg}) \frac{dP_{32}^c}{ds_3}(C_3) C_3',
$$

- Use (1) to determine of $P_{cg}$ on $\partial \mathbb{T}$:
  - on the water-gas side $\Rightarrow P_{13}^{cg}$
  - on the water-oil side $\Rightarrow P_{12}^{cg}$
  - on the gas-oil side $\Rightarrow P_{23}^{cg}$

- TD-compatibility condition for the two-phase $k_r$ data:
  $$P_{13}^{cg}(1) = P_{23}^{cg}(1) (= +\infty !)$$
Honoring two-phase data: 1 Determine $P_{cg}$ on $\partial T$

- Adjust $P_{cg}^{32}$ to satisfy the TD-compatibility condition $P_{cg}^{13}(1) = P_{cg}^{32}(1)$
Honoring two-phase data: 1 Determine $P_{cg}$ on $\partial \mathbb{T}$

- Adjust $P_{cg}^{32}$ to satisfy the TD-compatibility condition $P_{cg}^{13}(1) = P_{cg}^{32}(1)$
- Implies minor changes on the oil-gas relative permeabilities:
Honoring two-phase data: 1 Determine $P_{cg}$ on $\partial \mathcal{T}$

- now $P_{cg}$ is available on $\partial \mathcal{T}$!
Honoring two-phase data: determine $\partial P_{cg}/\partial n$ on $\partial \mathbb{T}$

- Let $p$ be a given global pressure level.
- Let $s \in \partial \mathbb{T}$, $n$ normal to $\partial \mathbb{T}$ at $s$, and define:
  \[ C : t \in [0, \epsilon] \mapsto C(t) = s - tn \in \mathbb{T} \]
  a curve normal to $\partial \mathbb{T}$ at $s$.

Then
\[
\frac{\partial P_{cg}}{\partial n}(s) = -\frac{d}{dt} P_{cg}(C(t))|_{t=0}.
\]
Honoring two-phase data: determine $\partial P_{cg}/\partial n$ on $\partial \mathbb{T}$

- Let $p$ be a given global pressure level.
- Let $s \in \partial \mathbb{T}$, $n$ normal to $\partial \mathbb{T}$ at $s$, and define:

  $$C : t \in [0, \epsilon] \rightarrow C(t) = s - tn \in \mathbb{T}$$

  a curve normal to $\partial \mathbb{T}$ at $s$.

Then

$$\frac{\partial P_{cg}}{\partial n}(s) = -\frac{d}{dt} P_{cg}(C(t)) \bigg|_{t=0}.$$

But using again equation (1):

$$\frac{dP_{cg}}{dt} = f_1(C, p - P_{cg}) \frac{dP_{c}^{12}}{ds_1}(C_1) C'_1 + f_3(C, p - P_{cg}) \frac{dP_{c}^{32}}{ds_3}(C_3) C'_3,$$

gives:

$$\frac{\partial P_{cg}}{\partial n} = \begin{cases} 
\frac{\sqrt{3}}{3} f_{12} \frac{dP_{c}^{12}}{ds_1} & \text{(water-oil edge)}, \\
\frac{\sqrt{3}}{3} (f_{13} \frac{dP_{c}^{12}}{ds_1} + f_{33} \frac{dP_{c}^{32}}{ds_3}) & \text{(water-gas edge)}, \\
\frac{\sqrt{3}}{3} f_{33} \frac{dP_{c}^{32}}{ds_3} & \text{(gas oil edge)}.
\end{cases}$$
Honoring two-phase data: determine mobility $d$ on $\partial\Omega$

$$d = \begin{cases} 
kr_1 d_1 (p - P_{cg}^{12} + P_c^{12}) + kr_2 d_2 (p - P_{cg}^{12}) & \text{(water-oil)} \\
kr_1 d_1 (p - P_{cg}^{13} + P_c^{12}) + kr_3 d_3 (p - P_{cg}^{13} + P_c^{32}) & \text{(gas-water)} \\
kr_3 d_3 (p - P_{cg}^{32} + P_c^{32}) + kr_2^{32} d_2 (p - P_{cg}^{32}) & \text{(gas-oil)} 
\end{cases}$$
Honoring two-phase data: determine mobility \( d \) on \( \partial T \)

\[
d = \begin{cases} 
kr_1 d_1 (p - P_{cg}^{12} + P_c^{12}) + kr_2 d_2 (p - P_{cg}^{12}) & \text{(water-oil)} \\
kr_1 d_1 (p - P_{cg}^{13} + P_c^{12}) + kr_3 d_3 (p - P_{cg}^{13} + P_c^{32}) & \text{(gas-water)} \\
k r_3 d_3 (p - P_{cg}^{32} + P_c^{32}) + kr_2^{32} d_2 (p - P_{cg}^{32}) & \text{(gas-oil)}
\end{cases}
\]
Honoring two-phase data: 4 - Find $P_{cg}$ and $d$ inside $\mathbb{T}$

A - FIRST APPROACH: INTERPOLATION

- Harmonic for $d$ and Biharmonic for $P_{cg}$:

\[
\begin{aligned}
-\Delta d &= 0 \quad \text{in } \mathbb{T}, \\
\frac{\partial d}{\partial n} &= d^{\text{data}} \quad \text{on } \partial\mathbb{T}.
\end{aligned}
\]

\[
\begin{aligned}
\Delta^2 P_{cg} &= 0 \quad \text{in } \mathbb{T}, \\
\frac{\partial P_{cg}}{\partial n} &= \frac{P_{cg}^{\text{data}}}{\partial n} \quad \text{on } \partial\mathbb{T}.
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\Delta^2 P_{cg} &= 0 \quad \text{in } \mathbb{T}, \\
\quad P_{cg} &= P_{cg_{\text{data}}} \quad \text{on } \partial\mathbb{T}, \\
\quad \frac{\partial P_{cg}}{\partial n} &= \frac{\partial P_{cg_{\text{data}}}}{\partial n} \quad \text{on } \partial\mathbb{T}.
\end{cases}$$

- Finite element parameterization:

  reduced HCT for $P_{cg}$, $P^1$ for $d$.

  (di Chiara Roupert, Chavent and Schaefer, J. Comp. Physic 2010)
Honoring two-phase data: 4 - Find $P_{cg}$ and $d$ inside $\mathcal{T}$

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\end{align*}
$$

$$
\begin{align*}
\Delta^2 P_{cg} &= 0 \quad \text{in } \mathcal{T},

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\quad \frac{\partial P_{cg}}{\partial n} &= \frac{\partial P_{cg_{\text{data}}}}{\partial n} \quad \text{on } \partial \mathcal{T}.
\end{align*}
$$

- Finite element parameterization:

  reduced HCT for $P_{cg}$, $P^1$ for $d$.

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- Limitations: no control on relative permeabilities inside $\mathcal{T}$

**B - SECOND APPROACH : OPTIMIZATION** ...still to be tested!
INTERPOLATION - Numerical results:

- global mobility $d$:

![Graph showing numerical results with axes and annotations]

Happy retirement Jean and Jérôme!
INTERPOLATION - Numerical results:

- global capillary pressure \( P_{cg} = \text{global pressure } p - \text{oil pressure } p_2 \)
INTERPOLATION - Numerical results:

- global pressure $p$ - water pressure $p_1 = P_{cg} - P_c^{12}$
INTERPOLATION - Numerical results:

- **global pressure** $p$ - **gas pressure** $p_3 = P_{cg} - p_{c}^{32}$
Step 1: $P_{cg} \Rightarrow$ TD Three-phase fractional flows

$$\nu_j = \frac{\partial P_{cg}}{\partial s_j} \left/ \frac{dP_c^{ij2}}{ds_j} \right., \ j = 1, 3 \ , \ \nu_2 = 1 - \nu_1 - \nu_3$$
INTERPOLATION - Back to relative permeabilities:

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INTERPOLATION - Back to relative permeabilities:

- **Step 2**: \( P_{cg}, d \) ⇒ TD Three-phase relative permeabilities
**INTERPOLATION - Back to relative permeabilities:**

- **Step 2**: \( P_{cg}, d \Rightarrow \text{TD Three-phase relative permeabilities} \)

![Diagram showing relative permeabilities](image-url)
INTERPOLATION - Back to relative permeabilities:

- Step 2: $P_{cg}, d \Rightarrow$ TD Three-phase relative permeabilities

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For three-phase compressible flows:

- Equivalent global pressure formulation: no gradient of capillary pressure in pressure equation!
- Well defined transformation ($0 \leq \partial P_{cg}/\partial p < 1$)
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  remains smooth when the mobility of one fluid vanishes
Conclusion

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Bad news: three-phase data have to satisfy a TD condition
Good news: TD relative permeabilities can be INTERPOLATED
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Next step: replace INTERPOLATION by OPTIMIZATION:

• take constraints into account,
• try to match \(kr^\text{target}_j(s), j = 1 \ldots 3\).